Lecture 3, Sep 9, 2022

Formal Definition of the Double Integral

Rectangular Regions

Definition

Let $z = f(x, y)$ be defined on $R = \{\,(x, y) \mid a \le x \le b, c \le y \le d\,\}$, then the double integral of f over R is

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i$$

where:

- The region R is divided into N rectangular regions, with region i having dimensions Δx_i by Δy_i, and area $\Delta A_i = \Delta x_i \Delta y_i$
- (x_i^*, y_i^*) is a point in region i
- Δd_i is the length of the diagonal of region i
- The norm of the partition $\|P\| = \max(\Delta d_i)$ for $i = 1, 2, \cdots, N$

- $\displaystyle\sum_{i=1}^{N} m_i \Delta x_i \Delta y_i \le \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta x_i \Delta y_i \le \sum_{i=1}^{N} M_i \Delta x_i \Delta y_i$ where m_i/M_i is the minimum/maximum value of f over region i
  - Convergence is guaranteed by $m_i \to M_i$ as $\|P\| \to 0$, which is guaranteed by the continuity of f over R
- If f is continuous over R, this limit always exists and is the same for any way of dividing and sampling R, and f is said to be integrable over R

Definition

Double sum definition:

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_i$$

where:

- The region R is divided into an n by m grid of rectangular regions, with region ij being Δx_i by Δy_j, with an area of $\Delta A_{ij} = \Delta x_i \Delta y_j$
- (x_{ij}^*, y_{ij}^*) is a point in region ij
- Δd_{ij} is the length of the diagonal of region ij
- The norm of the partition $\|P\| = \max(\Delta d_{ij})$ for $\begin{cases} i = 1, 2, \cdots, n \\ j = 1, 2, \cdots, m \end{cases}$

Non-Rectangular Regions

Definition

Let $f(x, y)$ be defined and rectangular on a region R, then:

- Divide the region into N rectangular regions such that the regions are completely within R (some of R is omitted at the boundaries), then $\sum_{i=1}^{N} \Delta A_i \leq A$ and $\sum_{i=1}^{N} m_i \Delta A_i \leq V$ where A is the actual area of R, V is the actual volume
- Divide the region into M rectangular regions such that the R is completely covered (some regions extend past R at the boundaries), then $\sum_{i=1}^{N} \Delta A_j \geq A$ and $\sum_{j=1}^{M} M_j \Delta A_j \geq V$

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i = \lim_{\|P\|\to 0} \sum_{j=1}^{M} f(x_j^*, y_j^*)\Delta A_j$$

A double sum can also be used, then $\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{j=1}^{M} \sum_{i=1}^{N} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$