Lecture 24, Oct 28, 2022

Pressure Variations in a Fluid at Rest

- $\vec{a} = 0 \implies \vec{\nabla} p = -\rho g \vec{k} \implies \left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k} \right) = -\rho g \vec{k}$
  - This means $\begin{cases} \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial y} = 0 \\ \frac{\partial p}{\partial z} = -\rho g \end{cases}$
- This means in a fluid at rest $p = p(z)$
  - Pressure is constant along the x and y directions
  - Note the minus sign; as z decreases the pressure goes up
- We can now integrate this: $\Delta p = \int_{z_1}^{z_2} \rho g \, \mathrm{d}z = g \int_{z_1}^{z_2} \rho \, \mathrm{d}z$
  - Case 1: incompressible fluid, ρ constant
    * $\Delta p = -\rho g(z_2 - z_1)$
    * We can write this as $p_2 = p_1 - \rho g h$ or $p_1 = p_2 + \rho g h$ where $h = z_2 - z_1$
      - Note if we're moving down in the fluid, $h < 0$
    * In this case pressure varies linearly with z
  - Case 2: compressible fluid, $\rho = \rho(z)$
    * Define the *specific weight* γ as the weight per unit volume
    * $\gamma = \frac{W}{V} = \frac{mg}{V} = \frac{\rho V g}{V} = \rho g$
    * Therefore the change in pressure is found by integrating the specific weight
    * Note $\rho_{\text{gas}} \ll \rho_{\text{liquid}}$ so by extension, $\left| \frac{\partial p}{\partial z}_{\text{gas}} \right| \ll \left| \frac{\partial p}{\partial z}_{\text{liquid}} \right|$
      - Pressure changes are much smaller in a gas
    * For small distances (e.g. within a tube or a building) in a gas, we can assume pressure to be constant

Summary

For a fluid with no shear force:

$$-\vec{\nabla} p + \rho \vec{g} = \rho \vec{a}$$

If gravity is $\vec{g} = -g\hat{k}$:

$$-\vec{\nabla} p - \rho g \vec{k} = \rho \vec{a}$$

If the fluid is at rest:

$$\Delta p = \int_{z_1}^{z_2} \rho g \, \mathrm{d}z = g \int_{z_1}^{z_2} \rho \, \mathrm{d}z$$

If the fluid is also incompressible:

$$p_2 = p_1 - \rho g h$$

- Example: find the pressure elevation relationship for a perfect ideal gas
  - $p = \rho R T$
  - $\frac{\partial p}{\partial z} = -\rho g \implies \frac{\partial p}{\partial z} = -\frac{p}{RT} g$
  - $\int_{p_1}^{p_2} \frac{1}{p} \, \mathrm{d}p = -\int_{z_1}^{z_2} \frac{g}{RT} \, \mathrm{d}z \implies \ln \frac{p_2}{p_1} = -\frac{g}{RT} \Delta z$
  - $p_2 = p_1 e^{-\frac{g}{RT}(z_2 - z_1)}$