Lecture 23, Oct 28, 2022

Hydrostatics

- Hydrostatics deal with fluids at rest or in rigid body motion (e.g. holding a cup and moving)
  - The fluid particles do not move relative to each other
- Recall there are 2 categories of forces on a fluid, body forces and surface forces
  - Surface forces can be split into normal forces (pressure) and shear forces
  - In hydrostatics, since fluid particles don't move relative to each other, there are no shear forces

Pressure

- A normal surface force acting on fluids
- The result of molecular collisions on a real or imaginary surface
  - Molecules colliding with the surface undergo a momentum change, which acts on the surface with a force
- A flat surface may be rough at a microscopic scale, but statistically any tangential forces generated will be cancelled out, so total pressure is normal to the surface
- Pressure is the inward acting normal force per unit area (i.e. normal stress)
- $\mathrm{d}\vec{F} = -p\,\mathrm{d}A\vec{n}$
  - The force vector is proportional to area, and in the opposite direction as the outward facing normal vector
  - We can get a large p with a small F if A is small (e.g. ice skating melting ice)

Pressure at a Point

- Consider a point in a static fluid and an imaginary plane that goes through the point; we can take the limit as the area around the point goes to 0 to define the pressure stress
- There are however infinitely many planes that go through this point, but we can prove that the pressure is equal no matter what direction

Important

Pascal's Law: Pressure at a point in hydrostatic fluid is the same in all directions

- Proof:
  - Consider a container with fluid undergoing rigid body motion
  - Consider a triangular wedge within the container; we have body forces (gravity) and surface forces (pressure) acting on it
    - This includes a slanted plane with some angle θ, sides $\delta x, \delta y, \delta z$ and diagonal δs
  - Pressure forces on the region:
    - On the bottom: $p_z \delta y \delta x$
    - Similarly on the left: $p_y \delta y \delta x$
    - On the slanted surface: $p_s \delta s \delta x$
    - Note we can ignore the pressure forces on the front and back for now
    - Gravitational force $mg = \rho g \left(\dfrac{\delta x \delta y \delta z}{2}\right)$
  - Force balance: $\begin{cases} \sum F_y = ma_y = p_y \delta z \delta x - p_s \delta s \delta x \sin\theta = \rho \dfrac{\delta x \delta y \delta z}{2} a_y \\ \sum F_z = ma_z = p_z \delta y \delta x - p_s \delta s \delta x \cos\theta - \rho g \left(\dfrac{\delta x \delta y \delta z}{2}\right) = \rho \dfrac{\delta x \delta y \delta z}{2} a_z \end{cases}$
    - Note $\delta s = \dfrac{\delta y}{\cos\theta} = \dfrac{\delta z}{\sin\theta}$

    * Substitute and cancel: $\begin{cases} p_y - p_s = \rho \frac{\delta y}{2} a_y \\ p_z - p_s = \rho \frac{\delta z}{2} (a_z + g) \end{cases}$
  - Now take the limit as the size of the region approaches 0 without changing its shape:
    * Let $\delta y \to 0$ then from the first equation we have $p_y = p_s$
    * Let $\delta z \to 0$ then from the second equation we have $p_z = p_s$
    * Therefore $p_s = p_y = p_z$
  - Since these directions are arbitrary, we know pressure is the same for any direction

Basic Pressure Field Equation

- This time, consider a small cubic region inside the fluid with sides $\delta x, \delta y, \delta z$, with center (x_0, y_0, z_0)
- Note: a free surface is denoted in fluid mechanics diagrams with a floating triangular thing
- We want a pressure function $p = p(x, y, z)$
- We again have gravitational and pressure forces
  - We can use a linear approximation to find the pressure at the edge of the region from the pressure in the center
    * Pressure on the left: $p_L = p - \frac{\partial p}{\partial y} \frac{\delta y}{2}$
    * Pressure on the right: $p_L = p + \frac{\partial p}{\partial y} \frac{\delta y}{2}$
    * Similarly for the 4 other faces
  - Now find the pressure forces:
    * Pressure force on the left would be $p_L \delta x \delta z = \left(p - \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z$
    * Pressure force on the right: $\left(p_L = p + \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z$
    * Repeat for the other 4 faces
  - Gravitational force: $\rho g \delta x \delta y \delta z$
  - $\sum \delta F_{s,y} = \left(p - \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z - \left(p_L = p + \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z$
    * $\sum \delta \vec{F}_s = - \left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \right) \delta x \delta y \delta z$
  - Total surface force per unit volume is then $-\vec{\nabla} p$
  - Total body force per unit volume is $-\rho g \hat{k}$
- Apply Newton's Second Law:
  - $\sum \delta \vec{F} = \delta m \vec{a}$
  - $-\vec{\nabla} p \delta x \delta y \delta z - \rho g \delta x \delta y \delta z \hat{k} = (\rho \delta x \delta y \delta z) \vec{a}$
  - $-\delta p - \rho g \hat{k} = \rho \vec{a}$
  - Net pressure force per unit volume plus net body force per unit volume is equal to mass times acceleration per unit volume

Equation

General equation of motion for fluids in which there are no shear forces:

$$-\delta p - \rho g \hat{k} = \rho \vec{a}$$

or

$$-\vec{\nabla} p + \rho \vec{g} = \rho \vec{a}$$

if $\vec{g}$ is not in the negative $\hat{k}$ direction