Lecture 20, Oct 21, 2022

Divergence

Definition

Let $\vec{F} = \vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$ be a differentiable vector field, then the divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(P\hat{i} + Q\hat{j} + R\hat{k}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- The divergence is a scalar field
- $\vec{\nabla} \cdot \vec{F}$ is a measure of the local "outgoingness" of the vector field at each point
  - If $\vec{\nabla} \cdot \vec{F} < 0$, the point is a sink – vectors around it go towards it
    - If $\vec{F}$ is a velocity field, with time particles accumulate at this point, increasing the density with time
  - If $\vec{\nabla} \cdot \vec{F} > 0$, then point is a source – vectors around it move away from i
    - Likewise particles move away from it and decrease the density with time
  - If $\vec{\nabla} \cdot \vec{F} = 0$, the density of the fluid around that point does not change
    - In an incompressible fluid, this is one of the governing equations

Curl

Definition

Let $\vec{F} = \vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$ be a differentiable vector field, then the curl of $\vec{F}$ is

$$\operatorname{curl}\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

- Unlike divergence, curl is a vector field
- The curl vector is associated with the local rotation of the vector field
  - The direction of the vector is the axis of rotation, the magnitude of the vector is the velocity of rotation
- $\vec{\nabla} \times \vec{V}$ where $\vec{V}$ is a velocity field is the vorticity

Properties of Divergence and Curl

- Let $f = f(x, y, z)$ be a scalar function and $\vec{F} = \vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$ be a vector field, both with continuous first partials, then:
  1. $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ (curl of a gradient is zero)
  2. If $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = 0$
     - This essentially comes from the first property; if $\vec{F}$ is conservative then it's a gradient of some scalar function, and we know the curl of a gradient is zero
  3. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$, (divergence of curl is zero)
- $\operatorname{div}(\vec{\nabla} \cdot f) = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ is known as the Laplace operator

Stokes' Theorem

- Stokes' theorem is the 3D extension of Green's theorem
  - Green's theorem connects the line integral around a closed *planar* curve C and the region R inside
  - Stokes' theorem does this in 3 dimensions, where the curve and its enclosed region is not necessarily planar
    * Think about a bubble surface made by a wire loop

Theorem

Let S be an orientable, piecewise smooth surface that is bound by a simple, closed, piecewise smooth and positively oriented boundary curve C; if $\vec{F}$ is a vector field with continuous first partials over S, then

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, \mathrm{d}S$$

- Stokes' theorem relates the line integral around the boundary of a surface to the surface integral of the curl over that surface
- The orientations of $\vec{n}$ and C must match the right-hand rule: curl your fingers in the direction of C, and the direction your thumb is pointing is the direction of positive $\vec{n}$
  - If you walk along the curve with your head in the direction of $\vec{n}$, if the surface is on your left, then the orientations match
- Green's theorem can be derived from Stokes' theorem:
  - For a surface S in the x-y plane with boundary curve C, its normal vector will be $\vec{n} = \vec{k}$
  - Therefore $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{k} \, \mathrm{d}S$
  - This works out to be equivalent to Green's theorem
- Note the surface of a boundary curve is not unique; we can stretch the surface however we want, as long as the boundary curve stays the same, the surface integral of the curl stays the same

Divergence Theorem

Theorem

Let E be a solid region bounded by the closed, positively (outward) oriented surface S; let $\vec{F}$ be a vector field with continuous first partials in E, then

$$\iint_S \vec{F} \cdot \vec{n} \, \mathrm{d}S = \iiint_E (\vec{\nabla} \cdot \vec{F}) \, \mathrm{d}V$$

- The flux across the boundary of a solid region is equal to the divergence inside the region