Lecture 19, Oct 20, 2022

Surface Integrals of Vector Fields

- Orientation starts to matter when we talk about vector functions!
  - Some surfaces are orientable, that is, two sided
  - Some surfaces are non-orientable, e.g. a Möbius strip
- A surface has two normal vectors; for an orientable surface, we can define a positive side and a negative side
  - For an open surface, the sign convention is arbitrary unless specified
  - For a closed surface, the convention is always outside is positive
- Consider $S : \vec{r}(u, v)$, a normal to the surface is $\vec{N} = \vec{r}_u \times \vec{r}_v$; the unit normal is $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$
  - If the surface is $z = f(x, y)$, then $\vec{n} = \dfrac{-\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$
- Consider some imaginary, smooth, orientable surface and some fluid with density $\rho(x, y, z)$ and velocity $\vec{V}(x, y, z)$; we can use a surface integral over the vector field to find out how much mass passes through this surface per unit time (the flux)
  - Divide the surface into subregions; for a subregion S_i, the volume of fluid passing through per unit time is $\vec{V} \cdot \vec{n}$
  - Therefore the mass flow through the surface is $\iint_S \rho(x, y, z)\vec{V}(x, y, z) \cdot \vec{n}\,\mathrm{d}S = \iint_S \vec{F} \cdot \vec{n}\,\mathrm{d}S = \iint_S \vec{F} \cdot \mathrm{d}\vec{S}$
  - This is known as *flux integration*
  - Note $\mathrm{d}S = \|\vec{r}_u \times \vec{r}_v\|\,\mathrm{d}u\,\mathrm{d}v$ and $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, so $\vec{n}\,\mathrm{d}S = \mathrm{d}\vec{S} = \vec{r}_u \times \vec{r}_v$

Definition

The flux of the vector field $\vec{F}(x, y, z)$ over a surface $S : \vec{r}(u, v)$ is given by

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S \vec{F} \cdot \vec{n}\,\mathrm{d}S = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\,\mathrm{d}u\,\mathrm{d}v$$

- For a closed surface, the flux is positive if there is a net outflow through the surface