Lecture 18, Oct 14, 2022

Parametric Surfaces

- We can parametrize a surface as $\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}, (u, v) \in D$
  - We need 2 parameters since a surface is a 2-dimensional construct
- The simplest way of doing so is to find $z = f(x, y)$, then parameterize $x = u, y = v \implies S = u\hat{i} + v\hat{j} + f(u, v)\hat{k}$
  - This is not always easy or possible to do
- A surface can have multiple parameterizations
- If we fix one parameter, we get a function of the other parameter which is a 1D curve slice of the surface
  - Let $P = (u_0, v_0)$
  - $\left.\dfrac{\partial \vec{r}(u, v)}{\partial v}\right|_{(u_0, v_0)} = \vec{r}_v(u_0, v_0)$ is a tangent vector in the direction of v at P
  - $\left.\dfrac{\partial \vec{r}(u, v)}{\partial u}\right|_{(u_0, v_0)} = \vec{r}_u(u_0, v_0)$ is a tangent vector in the direction of u at P
  - If we assume $\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0) \neq \vec{0}$ at this point, then this cross product is normal to the surface

Definition

A surface is smooth if for every point on S, $\vec{r}_u \times \vec{r}_v \neq \vec{0}$

- The local tangent plane will be normal to $\vec{r}_u \times \vec{r}_v$ and contains the point P; using this we can obtain a tangent plane

Definition

The surface area of a parametric surface can be found by

$$S = \iint_D \|\vec{r}_u \times \vec{r}_v\| \,\mathrm{d}u\,\mathrm{d}v$$

where $S : \vec{r}(u, v), (u, v) \in D$

- Note that in the special case where $\begin{cases} x = x \\ y = y \\ z = f(x, y) \end{cases}$, this just reduces to $\iint_D \sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}\,\mathrm{d}x\,\mathrm{d}y$, which we have seen before

Surface Integrals

- We can extend integration over planar surfaces to any general surface with a surface integral

Definition

The surface integral of a continuous scalar function $f(x, y, z)$ over a smooth, parametrized surface $S : \vec{r}(u, v), (u, v) \in D$ is given by

$$\iint_S f(x, y, z)\,\mathrm{d}S = \iint_D f(x(u, v), y(u, v), z(u, v))\|\vec{r}_u \times \vec{r}_v\|\,\mathrm{d}u\,\mathrm{d}v$$

- We can also sum together multiple piecewise smooth surfaces