Lecture 17, Oct 14, 2022

Green's Theorem

- Green's theorem connects a line integral over a closed curve C to a double integral over the region R it encloses
- Simple curve: a curve that does not intersect itself except at the endpoints
- Curves can have orientation; a positive oriented curve has the inside of the curve to the left as you go around it (think unit circle)

Theorem

For a positively oriented, piecewise smooth, simple closed curve C in 2D bounding the region R, if $P(x,y), Q(x,y)$ and their first partials are continuous over R, then

$$\oint P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Green's Theorem: Proof of Special Case

- Consider a region that can be expressed as both type 1 and 2: $R = \begin{cases} \{\,(x,y) \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\,\} \\ \{\,(x,y) \mid x_1(y) \leq x \leq x_2(y), c \leq y \leq d\,\} \end{cases}$
- For the type 1 form, $$\begin{aligned}\oint P(x,y)\,\mathrm{d}x &= \int_{C_1} P(x,y)\,\mathrm{d}x + \int_{C_2} P(x,y)\,\mathrm{d}x \\ &= \int_a^b P(x,y_1(x))\,\mathrm{d}x - \int_a^b P(x,y_2(x))\,\mathrm{d}x \\ &= -\int_a^b \left(P(x,y_2(x)) - P(x,y_1(x))\right)\mathrm{d}x \\ &= -\int_a^b \left[P(x,y)\right]_{y=y_1(x)}^{y=y_2(x)}\,\mathrm{d}x \\ &= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\,\mathrm{d}y\,\mathrm{d}x \\ &= -\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}A\end{aligned}$$
  - C_1 is the bottom path $y = y_1(x)$, C_2 is the top path $y = y_2(x)$
- For the type 2 form, $$\begin{aligned}\oint Q(x,y)\,\mathrm{d}y &= \int_{C_3} Q(x,y)\,\mathrm{d}y + \int_{C_4} Q(x,y)\,\mathrm{d}y \\ &= -\int_c^d Q(x_1(y),y)\,\mathrm{d}y + \int_c^d Q(x_2(y),y)\,\mathrm{d}y \\ &= \int_c^d \left(Q(x_2(y),y) - Q(x_1(y),y)\right)\mathrm{d}y \\ &= \int_c^d \left[Q(x,y)\right]_{x=x_2(y)}^{x=x_1(y)}\,\mathrm{d}y \\ &= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_R \frac{\partial Q}{\partial x}\,\mathrm{d}A\end{aligned}$$
  - C_3 is the left path $x = x_1(y)$, C_4 is the right path $x = x_2(y)$
- If we combine the two, we get: $\oint_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A$