Lecture 16, Oct 13, 2022

Fundamental Theorem for Line Integrals

- Recall that the fundamental theorem for single integrals is $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$

Definition

A vector field $\vec{F}$ is conservative if $\vec{F} = \vec{\nabla} f$, i.e. it is the gradient of some scalar function f

The scalar function f is known as the potential function of $\vec{F}$

- Suppose $\vec{F}$ is conservative and let C be a smooth curve given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, t \in [a, b]$
- $\int_C \vec{F}(x, y, z)\,\mathrm{d}\vec{r} = \int_C \vec{\nabla} f(x, y, z)\,\mathrm{d}\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t$
  - Notice $f(\vec{r}(t)) \cdot \vec{r}'(t) = \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k}\right)$
  - From the chain rule, this is equal to $\frac{\mathrm{d}f}{\mathrm{d}t}$
  - Therefore $\int_C \vec{F}(x, y, z)\,\mathrm{d}\vec{r} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\vec{r}(t))\,\mathrm{d}t = f(\vec{r}(b)) - f(\vec{r}(a))$

Theorem

Fundamental theorem for line integrals:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{\nabla} f \cdot \mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

As a consequence $\vec{F} = \vec{\nabla} f \implies \oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C \vec{\nabla} f \cdot \mathrm{d}\vec{r} = 0$ where C is any closed curve

- For a conservative vector field, line integrals are path independent
- This also works in reverse; if $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$ for all piecewise smooth closed curves C, then $\vec{F}$ is conservative

Determining Conservativeness

- If a vector field is conservative, then $\vec{F}(x, y) P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath} = \vec{\nabla} f$
- Therefore $P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}$, so by Clairaut's Theorem $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \implies \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Important

A two variable function $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

For a three-variable function $\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + S(x,y,z)\hat{k}$, the requirement is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial S}{\partial y}, \frac{\partial P}{\partial z} = \frac{\partial S}{\partial x}$$

- For multiple variables, all mixed partials have to be equal