Lecture 15, Oct 7, 2022

Line Integrals of Vector Fields

- A vector field is $\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$
  - For each (x,y,z) in space the vector field associates it with a vector
  - We can also express this as $\vec{F}(x,y,z) = \vec{F}(\vec{r})$
- Once again we assume C is smooth
- Consider the example of computing the work done to a particle in a force field
  - Divide the curve into many segments of Δs_i
  - $W_i = \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*)\Delta s_i$
  - $\vec{T}$ is the unit tangent vector; taking the dot product gets us the amount of force in the direction of movement
  - $W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,\mathrm{d}s$
- Parametrize $C : \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, t \in [a,b]$, then the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$
- $W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,\mathrm{d}s = \int_a^b \vec{F}(x(t),y(t),z(t)) \cdot \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}\|\vec{r}'(t)\|\,\mathrm{d}t = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t$

Important

To compute the line integral of a vector field $\vec{F}(\vec{r})$ along C:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t$$

where $\mathrm{d}\vec{r} = \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \vec{T}\,\mathrm{d}s$

- Alternatively:
$$\begin{aligned}\int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t \\ &= \int_a^b \left(P\hat{\imath} + Q\hat{\jmath} + R\hat{k}\right) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k}\right)\,\mathrm{d}t \\ &= \int_C P\,\mathrm{d}x + \int_C Q\,\mathrm{d}y + \int_C R\,\mathrm{d}z \\ &= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z\end{aligned}$$
- Note the result of a line integral through a vector field is a scalar

Important

Unlike a line integral through a scalar field, $\int_C \vec{F} \cdot \mathrm{d}\vec{r} = -\int_C \vec{F} \cdot \mathrm{d}\vec{r}$, i.e. the direction matters