Lecture 14, Oct 7, 2022

Computing Line Integrals

- To compute a line integral over a curve we need to parametrize C: $x = x(t), y = y(t)$ or $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$
- Assumptions:
  1. $f(x, y)$ is continuous over C
  2. C is smooth (cannot contain kinks, etc)
     - $\vec{r}'(t)$ is continuous
       * Disallows sharp direction changes
     - $\vec{r}'(t) \neq \vec{0}$ except possibly at the endpoints of C
       * If this happens, the position vector stops, and then can go in any direction, so this can make a kink
- Similar to arc length, $\mathrm{d}s = \|\vec{r}'(t)\|\,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$

Important

To compute the line integral of $f(x, y)$ along curve C:

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

where C is parametrized as

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \implies \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, t \in [a, b]$$

with the assumptions that $\vec{r}'(t)$ is continuous and $\vec{r}'(t)$ is nonzero except at the endpoints of C

- In the 3D and more general case $\int_C f(x, y, z)\,\mathrm{d}s = \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\,\mathrm{d}t$
- If C is not smooth but piecewise smooth (consisting of finite number of smooth segments), we can break the integral into pieces
  - If $C = C_1 \cup C_2$ then $\int_C f(x, y)\,\mathrm{d}s = \int_{C_1} f(x, y)\,\mathrm{d}s + \int_{C_2} f(x, y)\,\mathrm{d}s$
- Note: $\int_C f(x, y)\,\mathrm{d}s = \int_{-C} f(x, y)\,\mathrm{d}s$, i.e. the direction of the curve doesn't matter for line integration, sine $\mathrm{d}s > 0$ always
  - This is unlike the 1D case