Lecture 12, Sep 30, 2022

Change of Variables in Double Integrals Examples

- Example: Change a double integral from rectangular to polar coordinates
  - $x = r\cos\theta, y = r\sin\theta$
  - $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$
  - $\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\,\mathrm{d}\theta = \iint_S f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$

Change of Variables in Triple Integrals

Definition

The Jacobian for a 3D transformation $x = g(u,v,w), y = h(u,v,w), z = l(u,v,w)$ is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Important

Under a change of variables $x = g(u,v,w), y = h(u,v,w), z = l(u,v,w)$,

$$\iiint_R f(x,y,z)\,\mathrm{d}V = \iiint_S f(g(u,v,w), h(u,v,w), l(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

with the same assumptions as in the double integral case

Successive Transformations

- If $x = x(u,v), y = (u,v)$ and $u = u(s,t), v = v(s,t)$, we can transform from x, y to s, t
- $\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)}$

Back Transformations

- The Jacobian associated with the inverse of a transformation is the inverse of the Jacobian of that transformation
- $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$
- Knowing the backward transformation $u = f(x,y), v = g(x,y)$ we can calculate $\frac{\partial(u,v)}{\partial(x,y)}$ and use that to find $\frac{\partial(x,y)}{\partial(u,v)}$ without having to explicitly calculate $x(u,v), y(u,v)$