Lecture 11, Sep 30, 2022

Change of Variables in Multiple Integrals

- Consider the single variable case for $\int 2x\sqrt{x^2+1}\mathrm{d}x$; let $x = x(u) \implies x = \sqrt{u-1} \implies \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2\sqrt{u-1}} \implies \int 2x\sqrt{x^2+1}\mathrm{d}x = \int 2\sqrt{u-1}\sqrt{u}\frac{1}{2\sqrt{u-1}}\,\mathrm{d}u = \int \sqrt{u}\,\mathrm{d}u$
  - This transformation from x to a function of u is one-to-one
- Before we can do this for multiple integrals, what does $\mathrm{d}x = \frac{1}{2\sqrt{u-1}}\,\mathrm{d}u$ mean?
  - We converted $f(x)$ to $f(u)$, so that the corresponding points in each have the same value of f
  - This alone doesn't make the integrals equal because $\Delta x \neq \Delta u$ since Δu is stretched or compressed
  - For small Δu w have $\frac{\Delta x}{\Delta u} = \frac{\mathrm{d}x}{\mathrm{d}u} \implies \Delta x = \frac{\mathrm{d}x}{\mathrm{d}u}\Delta u$
  - $\frac{\mathrm{d}x}{\mathrm{d}u}$ can be viewed as the scaling factor between the two areas
- Consider $\iint_R f(x,y)\,\mathrm{d}A$, change variables $x = g(u,v), y = h(u,v)$ assuming g, h have continuous first partials and a 1-to-1 mapping
  - The distortion on each ΔA can be calculated using the Jacobian, the determinant of the local first derivative matrix

Definition

The Jacobian of a transformation

$$x_1 = x_1(u_1, u_2, \cdots, u_n), x_2 = x_2(u_1, u_2, \cdots, u_n), \cdots, x_n = x_n(u_1, u_2, \cdots, u_n)$$

is

$$J = \frac{\partial(x,y)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

For a 2D change of variables $x = g(u,v), y = h(u,v)$,

$$J = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$$

Important

Under a change of variables $x = g(u,v), y = h(u,v)$,

$$\iint_R f(x,y)\,\mathrm{d}A_R = \iint_S f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}A_S = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v$$

where R is on the x-y plane, S is on the u-v plane, assuming:

1. f is continuous
2. g and h have continuous first partials
3. The transformation is 1-to-1
4. The Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ is nonzero

Note the absolute value around the Jacobian.

- To do the transformation, rewrite x and y in terms of u and v, replace $\mathrm{d}A$ with $\dfrac{\partial(x,y)}{\partial(u,v)}\,\mathrm{d}u\,\mathrm{d}v$, and change the bounds of integration to reflect the new region