Lecture 10, Sep 29, 2022

Taylor Series and Approximations for Two Variable Functions

- For $f(x, y)$, define a parametric $F(t) = f(x_0 + t\Delta x, y_0 + t\Delta y)$ where (x_0, y_0) is the point around which to approximate
  - $F(0) = f(x_0, y_0)$
  - We want to find $F(1) = f(x_0 + \Delta x, y_0 + \Delta y)$
- By the chain rule $F'(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x + \frac{\partial f}{\partial y}\Delta y$
- The second derivative is $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}x}{\mathrm{d}t}\Delta x + \frac{\partial^2 f}{\partial y \partial x}\frac{\mathrm{d}y}{\mathrm{d}t}\Delta x + \frac{\partial^2 f}{\partial x \partial y}\frac{\mathrm{d}x}{\mathrm{d}t}\Delta y + \frac{\partial^2 f}{\partial y^2}\frac{\mathrm{d}y}{\mathrm{d}t}\Delta y = \frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2$
- This gives the approximations:
  - The first order approximation is then $f(x_0 + \Delta x, y_0 + \Delta y) + f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$ (tangent plane approximation)
  - The quadratic approximation is $f(x_0 + \Delta x, y_0 + \Delta y) + f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \frac{1}{2!}\left(f_{xx}(x_0, y_0)\Delta x^2 + 2f_{xy}(x_0, y_0)\Delta x \Delta y + f_{yy}(x_0, y_0)\Delta y^2\right)$
- In general the nth order derivatives work like a binomial expansion

Definition

The Taylor series expansion of $f(x_0 + \Delta x, y_0 + \Delta y)$ is

$$f(x_0 + \Delta x, y_0 + \Delta y) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\left.\frac{\partial^n f}{\partial x^{n-k}\partial y^k}\right|_{(x_0, y_0)}\Delta x^{n-k}\Delta y^k\right\}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Important

Third-degree Taylor series expansion of a two-variable function:

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) \approx \quad & f(x_0, y_0) \\ & + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\ & + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2\right) \\ & + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\Delta x^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}\Delta x^2 \Delta y + 3\frac{\partial^3 f}{\partial x \partial y^2}\Delta x \Delta y^2 + \frac{\partial^3 f}{\partial y^3}\Delta y^3\right) \end{aligned}$$

where all partials are evaluated at (x_0, y_0)

- The nth degree polynomial of $f(x, y)$ is a polynomial in $x - x_0$ and $y - y_0$ with terms up to the nth power
- Approximations become exact as $\sqrt{\Delta x^2 + \Delta y^2} \to 0$