Lecture 1, Sep 8, 2022

Partial Integration

Definition

A variable which is kept constant during integration is called a *parameter*

- The result of integration is a function of the parameter
- This type of integration is called a *partial integral*, since other variables other than the variable of integration are held constant
  - Opposite of partial differentiation
  - $\int_a^b f(x, y)\,\mathrm{d}x$ is a partial integral wrt x

Iterated Integrals

Theorem

Fubini's Theorem: $\int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y$

- The order of integration for a double integral can be switched ("you can slice a region either way")
  - The counterpart is Clairaut's Theorem (symmetry of second partial derivatives)

Note

In the special case where $f(x, y) = g(x)h(y)$,

$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b g(x)\,\mathrm{d}x \int_c^d h(y)\,\mathrm{d}y$$

Lecture 2, Sep 9, 2022

More General Regions

Definition

- Type 1 region: $R = \{\, (x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \,\}$
- Type 2 region: $R = \{\, (x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y) \,\}$

Where g_1, g_2, h_1, h_2 are continuous

- Type 1 regions: hard boundaries in x, continuous varying boundaries in y
  - $V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$
- Type 2 regions: hard boundaries in y, continuous varying boundaries in x
  - $V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$
- When the region of integration is neither, it can be cut up into type 1 and 2 regions
- When dealing with these, it's useful to first draw the planar region R
- Sometimes it can be much easier to integrate along one axis first, and then the other
  - e.g. $z = e^{x^2}$ over $y = x, 0 \leq x \leq 1$ is much easier to integrate along y first

Lecture 3, Sep 9, 2022

Formal Definition of the Double Integral

Rectangular Regions

Definition

Let $z = f(x, y)$ be defined on $R = \{\, (x, y) \mid a \leq x \leq b, c \leq y \leq d \,\}$, then the double integral of f over R is

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i$$

where:

- The region R is divided into N rectangular regions, with region i having dimensions Δx_i by Δy_i, and area $\Delta A_i = \Delta x_i \Delta y_i$
- (x_i^*, y_i^*) is a point in region i
- Δd_i is the length of the diagonal of region i
- The norm of the partition $\|P\| = \max(\Delta d_i)$ for $i = 1, 2, \cdots, N$

- $\sum_{i=1}^{N} m_i \Delta x_i \Delta y_i \leq \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta x_i \Delta y_i \leq \sum_{i=1}^{N} M_i \Delta x_i \Delta y_i$ where m_i/M_i is the minimum/maximum value of f over region i
  - Convergence is guaranteed by $m_i \to M_i$ as $\|P\| \to 0$, which is guaranteed by the continuity of f over R
- If f is continuous over R, this limit always exists and is the same for any way of dividing and sampling R, and f is said to be integrable over R

Definition

Double sum definition:

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\| \to 0} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_i$$

where:

- The region R is divided into an n by m grid of rectangular regions, with region ij being Δx_i by Δy_j, with an area of $\Delta A_{ij} = \Delta x_i \Delta y_j$
- (x_{ij}^*, y_{ij}^*) is a point in region ij
- Δd_{ij} is the length of the diagonal of region ij
- The norm of the partition $\|P\| = \max(\Delta d_{ij})$ for $\begin{cases} i = 1, 2, \cdots, n \\ j = 1, 2, \cdots, m \end{cases}$

Non-Rectangular Regions

Definition

Let $f(x,y)$ be defined and rectangular on a region R, then:

- Divide the region into N rectangular regions such that the regions are completely within R (some of R is omitted at the boundaries), then $\sum_{i=1}^{N} \Delta A_i \leq A$ and $\sum_{i=1}^{N} m_i \Delta A_i \leq V$ where A is the actual area of R, V is the actual volume
- Divide the region into M rectangular regions such that the R is completely covered (some regions extend past R at the boundaries), then $\sum_{i=1}^{N} \Delta A_j \geq A$ and $\sum_{j=1}^{M} M_j \Delta A_j \geq V$

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i = \lim_{\|P\|\to 0} \sum_{j=1}^{M} f(x_j^*, y_j^*)\Delta A_j$$

A double sum can also be used, then $\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\|P\|\to 0} \sum_{j=1}^{M} \sum_{i=1}^{N} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$

Lecture 4, Sep 15, 2022

Double Integrals in Polar Coordinates

Important

Given a region $R = \{\, (r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta \,\}$ and $f(x,y)$, then:

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r\,\mathrm{d}\theta$$

- With more complicated regions, the bounds may be functions of r or θ

Lecture 5, Sep 16, 2022

Applications of Double Integrals

- Consider a thin plate with density $\rho = \rho(x,y)$ (with dimensions of M/A)
  - We can find the plate's total mass with $m = \iint_R \rho(x,y)\,\mathrm{d}A$
- We can find the plate's centre of mass by $\bar{x} = \frac{1}{A}\iint_R x\rho(x,y)\,\mathrm{d}A, \bar{y} = \frac{1}{A}\iint_R y\rho(x,y)\,\mathrm{d}A$
- For centroids: $x_c = \frac{1}{A}\iint_R x\,\mathrm{d}A, y_c = \frac{1}{A}\iint_R y\,\mathrm{d}A$
  - Like centre of mass, but uniform $\rho = 1$
- Moment of inertia: recall $I = \sum_{i=1}^{n} m_i r_i^2$
  - As an integral this is $I_0 = \iint_R \rho(x,y)(x^2+y^2)\,\mathrm{d}A$ (note I_0 is the moment of inertia about the origin)

- Note this means we can separate $I_0 = \iint_R \rho(x,y)x^2 \, \mathrm{d}A + \iint_R \rho(x,y)y^2 \, \mathrm{d}A = I_x + I_y$

Lecture 6, Sep 16, 2022

Surface Area

- Given $z = f(x,y)$ on region R, how do we find its surface area?
- Divide the surface into many subregions S_i with area ΔS_i
- For each subregion draw a tangent plane, the elementary region has area ΔT_i (think of a disco ball)
- $S \approx \sum_{i=1}^{n} \Delta T_i \implies S = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta T_i \implies S = \iint_S \mathrm{d}T$
- Project subregion T_i with area ΔT_i down to R and get a rectangular subregion R_i with area $\Delta A_i = \Delta y_i \Delta x_i$
  - ΔT_i is a parallelogram
  - The two vectors that define this parallelogram are $\vec{a}_i, \vec{b}_i$
  - $\vec{a}_i$ has slope $f_x(x_i, y_i)$, $\vec{b}_i$ has slope $f_y(x_i, y_i) \implies \vec{a}_i = \begin{bmatrix} \Delta x_i \\ 0 \\ f_x(x_i, y_i)\Delta x_i \end{bmatrix}, \vec{b}_i = \begin{bmatrix} 0 \\ \Delta y_i \\ f_y(x_i, y_i)\Delta y_i \end{bmatrix}$
- Cross product to get area: $\vec{a}_i \times \vec{b}_i = \begin{bmatrix} -f_x(x_i, y_i) \\ -f_y(x_i, y_i) \\ 1 \end{bmatrix} \Delta x_i \Delta y_i$
  - $\Delta T_i = \|\vec{a}_i \times \vec{b}_i\| = \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\Delta x_i \Delta y_i$, and now we can integrate to get surface area

Important

The surface area of $z = f(x,y)$ on R is

$$S = \iint_R \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1} \, \mathrm{d}A$$

Lecture 7, Sep 22, 2022

Triple Integrals

- Many ideas from double integrals carry over
- Given a continuous $w = f(x,y,z)$ over a 3D region Q with volume V, break V into sub-volumes ΔV_i, choose any sample point within the region $P_i(x_i^*, y_i^*, z_i^*) \in \Delta V_i$, then
  - Define the lower sum $\sum_{i=1}^{n} m_i \Delta V_i$ and upper sum $\sum_{i=1}^{n} M_i \Delta V_i$
  - If the function is continuous then $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta V_i = \iiint_Q f(x,y,z) \, \mathrm{d}V$
  - Note $\|P\|$ is now the largest diameter of all subvolumes
- Sometimes only one integration sign is used
- In Cartesian coordinates, $\iiint_Q f(x,y,z) \, \mathrm{d}V = \iiint_Q f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$
- Suppose $f(x,y,z)$ is continuous over $Q = \{\, (x,y,z) \mid a \le x \le b, c \le y \le d, r \le z \le s \,\}$, then $\iiint_Q f(x,y,z) \, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$
  - As with the double integral case, the order of integration can be freely rearranged

- If $Q = \{ (x, y, z) \mid (x, y) \in R, g_1(x, y) \leq z \leq g_2(x, y) \}$, then $\iiint_Q f(x, y, z)\,\mathrm{d}V = \iint_R \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}A$

Lecture 8, Sep 23, 2022

Applications of Triple Integrals

- Finding mass: If $\rho = \rho(x, y, z)$ is the mass per unit volume over a region Q, then $m = \iiint_Q \rho(x, y, z)\,\mathrm{d}V$
- The centres of mass:
  - $M_{yz} = \iiint_Q x\rho(x, y, z)\,\mathrm{d}V = m\bar{x} \implies \bar{x} = \dfrac{M_{yz}}{m}$
  - $M_{xz} = \iiint_Q y\rho(x, y, z)\,\mathrm{d}V = m\bar{y} \implies \bar{y} = \dfrac{M_{xz}}{m}$
  - $M_{xy} = \iiint_Q z\rho(x, y, z)\,\mathrm{d}V = m\bar{z} \implies \bar{z} = \dfrac{M_{xy}}{m}$
- The centroids:
  - $x_c = \dfrac{1}{V} \iiint_Q x\,\mathrm{d}V$
  - $y_c = \dfrac{1}{V} \iiint_Q y\,\mathrm{d}V$
  - $z_c = \dfrac{1}{V} \iiint_Q z\,\mathrm{d}V$
- Moment of inertia:
  - $I = \iiint_Q \rho(x, y, z)(r(x, y, z))^2\,\mathrm{d}V$ where r is the distance to the axis of rotation

Lecture 9, Sep 23, 2022

Cylindrical Coordinates

- Uses triplets of (r, θ, z)
  - z is the distance from the $r\theta$ plane, same as Cartesian z
  - r, θ work like polar coordinates
- Conversion: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

Important

If $f(x, y, z)$ is continuous in

$$Q = \{ (x, y, z) \mid (x, y) \in R, u_1(x, y) \leq z \leq u_2(x, y) \}$$

where

$$R = \{ (r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta) \}$$

then

$$\begin{aligned} \iiint_Q f(x, y, z)\,\mathrm{d}V &= \iint_R \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}A \\ &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \end{aligned}$$

- Note in cylindrical coordinates, $\mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

Spherical Coordinates

- Uses triplets of (ρ, θ, ϕ)
  - ρ is the distance from the origin, always non-negative
  - ϕ is the angle from the z axis
    * ϕ is between 0 (straight up) and π (straight down)
  - θ is the angle from the x axis, in the xy plane
- Conversion: $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$
  - $\rho \sin\theta$ is r in the xy plane
- Constant ρ: sphere
- Constant θ: vertical plane
- Constant ϕ: cone
- Therefore in spherical coordinates $\mathrm{d}V = \rho^2 \sin\phi \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}\theta$

Important

Triple integration in spherical coordinates:

$$\iiint_Q f(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \iiint_{Q'} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \,\rho^2 \sin\phi \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}\theta$$

Lecture 10, Sep 29, 2022

Taylor Series and Approximations for Two Variable Functions

- For $f(x, y)$, define a parametric $F(t) = f(x_0 + t\Delta x, y_0 + t\Delta y)$ where (x_0, y_0) is the point around which to approximate
  - $F(0) = f(x_0, y_0)$
  - We want to find $F(1) = f(x_0 + \Delta x, y_0 + \Delta y)$
- By the chain rule $F'(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x + \frac{\partial f}{\partial y}\Delta y$
- The second derivative is $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}x}{\mathrm{d}t}\Delta x + \frac{\partial^2 f}{\partial y \partial x}\frac{\mathrm{d}y}{\mathrm{d}t}\Delta x + \frac{\partial^2 f}{\partial x \partial y}\frac{\mathrm{d}x}{\mathrm{d}t}\Delta y + \frac{\partial^2 f}{\partial y^2}\frac{\mathrm{d}y}{\mathrm{d}t}\Delta y = \frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2$
- This gives the approximations:
  - The first order approximation is then $f(x_0 + \Delta x, y_0 + \Delta y) + f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$ (tangent plane approximation)
  - The quadratic approximation is $f(x_0 + \Delta x, y_0 + \Delta y) + f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \frac{1}{2!}\left(f_{xx}(x_0, y_0)\Delta x^2 + 2f_{xy}(x_0, y_0)\Delta x \Delta y + f_{yy}(x_0, y_0)\Delta y^2\right)$
- In general the nth order derivatives work like a binomial expansion

Definition

The Taylor series expansion of $f(x_0+\Delta x, y_0+\Delta y)$ is

$$f(x_0+\Delta x, y_0+\Delta y) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\left.\frac{\partial^n f}{\partial x^{n-k}\partial y^k}\right|_{(x_0,y_0)}\Delta x^{n-k}\Delta y^k\right\}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Important

Third-degree Taylor series expansion of a two-variable function:

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) \approx \quad & f(x_0,y_0) \\ & + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\ & + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x\Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2\right) \\ & + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\Delta x^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}\Delta x^2\Delta y + 3\frac{\partial^3 f}{\partial x\partial y^2}\Delta x\Delta y^2 + \frac{\partial^3 f}{\partial y^3}\Delta y^3\right) \end{aligned}$$

where all partials are evaluated at (x_0, y_0)

- The nth degree polynomial of $f(x,y)$ is a polynomial in $x-x_0$ and $y-y_0$ with terms up to the nth power
- Approximations become exact as $\sqrt{\Delta x^2+\Delta y^2} \to 0$

Lecture 11, Sep 30, 2022

Change of Variables in Multiple Integrals

- Consider the single variable case for $\int 2x\sqrt{x^2+1}\mathrm{d}x$; let $x = x(u) \implies x = \sqrt{u-1} \implies \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2\sqrt{u-1}} \implies \int 2x\sqrt{x^2+1}\mathrm{d}x = \int 2\sqrt{u-1}\sqrt{u}\frac{1}{2\sqrt{u-1}}\,\mathrm{d}u = \int \sqrt{u}\,\mathrm{d}u$
  - This transformation from x to a function of u is one-to-one
- Before we can do this for multiple integrals, what does $\mathrm{d}x = \frac{1}{2\sqrt{u-1}}\,\mathrm{d}u$ mean?
  - We converted $f(x)$ to $f(u)$, so that the corresponding points in each have the same value of f
  - This alone doesn't make the integrals equal because $\Delta x \neq \Delta u$ since Δu is stretched or compressed
  - For small Δu w have $\frac{\Delta x}{\Delta u} = \frac{\mathrm{d}x}{\mathrm{d}u} \implies \Delta x = \frac{\mathrm{d}x}{\mathrm{d}u}\Delta u$
  - $\frac{\mathrm{d}x}{\mathrm{d}u}$ can be viewed as the scaling factor between the two areas
- Consider $\iint_R f(x,y)\,\mathrm{d}A$, change variables $x = g(u,v), y = h(u,v)$ assuming g, h have continuous first partials and a 1-to-1 mapping
  - The distortion on each ΔA can be calculated using the Jacobian, the determinant of the local first derivative matrix

Definition

The Jacobian of a transformation

$$x_1 = x_1(u_1, u_2, \cdots, u_n), x_2 = x_2(u_1, u_2, \cdots, u_n), \cdots, x_n = x_n(u_1, u_2, \cdots, u_n)$$

is

$$J = \frac{\partial(x, y)}{\partial(u, v)} \equiv \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

For a 2D change of variables $x = g(u, v), y = h(u, v)$,

$$J = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$$

Important

Under a change of variables $x = g(u, v), y = h(u, v)$,

$$\iint_R f(x, y)\,\mathrm{d}A_R = \iint_S f(u, v)\left|\frac{\partial(x, y)}{\partial(u, v)}\right|\mathrm{d}A_S = \iint_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right|\mathrm{d}u\,\mathrm{d}v$$

where R is on the x-y plane, S is on the u-v plane, assuming:

1. f is continuous
2. g and h have continuous first partials
3. The transformation is 1-to-1
4. The Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ is nonzero

Note the absolute value around the Jacobian.

- To do the transformation, rewrite x and y in terms of u and v, replace $\mathrm{d}A$ with $\frac{\partial(x, y)}{\partial(u, v)}\,\mathrm{d}u\,\mathrm{d}v$, and change the bounds of integration to reflect the new region

Lecture 12, Sep 30, 2022

Change of Variables in Double Integrals Examples

- Example: Change a double integral from rectangular to polar coordinates
  - $x = r\cos\theta, y = r\sin\theta$
  - $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$
  - $\iint_R f(x, y)\,\mathrm{d}A = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right|\mathrm{d}r\,\mathrm{d}\theta = \iint_S f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$

Change of Variables in Triple Integrals

Definition

The Jacobian for a 3D transformation $x = g(u, v, w), y = h(u, v, w), z = l(u, v, w)$ is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Important

Under a change of variables $x = g(u, v, w), y = h(u, v, w), z = l(u, v, w)$,

$$\iiint_R f(x, y, z)\,\mathrm{d}V = \iiint_S f(g(u, v, w), h(u, v, w), l(u, v, w))\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

with the same assumptions as in the double integral case

Successive Transformations

- If $x = x(u, v), y = (u, v)$ and $u = u(s, t), v = v(s, t)$, we can transform from x, y to s, t
- $\frac{\partial(x, y)}{\partial(s, t)} = \frac{\partial(x, y)}{\partial(u, v)}\frac{\partial(u, v)}{\partial(s, t)}$

Back Transformations

- The Jacobian associated with the inverse of a transformation is the inverse of the Jacobian of that transformation
- $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$
- Knowing the backward transformation $u = f(x, y), v = g(x, y)$ we can calculate $\frac{\partial(u, v)}{\partial(x, y)}$ and use that to find $\frac{\partial(x, y)}{\partial(u, v)}$ without having to explicitly calculate $x(u, v), y(u, v)$

Lecture 13, Oct 6, 2022

Line Integrals

- With a normal single integral we integrate over a coordinate axis (e.g. x axis); a line integral integrates over an arbitrary curve
- Defined just like a regular integral: $\int_C f(x, y)\,\mathrm{d}s = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$, where:
  - Divide the curve C into segments Δs_i
  - (x_i^*, y_i^*) is a point in the segment Δs_i
  - The norm of the partitioning $\|P\| = \max(\Delta s_i)$, i.e. the longest segment
- Geometrically, this is equivalent to the area of the surface between $f(x, y)$ and the x-y plane, along the path C (the "curtain" on curve C)
  - Note if we took $f(x, y) = 1$ we just get $\int_C \mathrm{d}s$ which is the arc length of C

Lecture 14, Oct 7, 2022

Computing Line Integrals

- To compute a line integral over a curve we need to parametrize C: $x = x(t), y = y(t)$ or $\vec{r}(t) = x(t)\hat{i}+y(t)\hat{j}$
- Assumptions:
  1. $f(x, y)$ is continuous over C
  2. C is smooth (cannot contain kinks, etc)
     - $\vec{r}'(t)$ is continuous
       * Disallows sharp direction changes
     - $\vec{r}'(t) \neq \vec{0}$ except possibly at the endpoints of C
       * If this happens, the position vector stops, and then can go in any direction, so this can make a kink
- Similar to arc length, $\mathrm{d}s = \|\vec{r}'(t)\|\,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$

Important

To compute the line integral of $f(x, y)$ along curve C:

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

where C is parametrized as

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \implies \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, t \in [a, b]$$

with the assumptions that $\vec{r}'(t)$ is continuous and $\vec{r}'(t)$ is nonzero except at the endpoints of C

- In the 3D and more general case $\int_C f(x, y, z)\,\mathrm{d}s = \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\,\mathrm{d}t$
- If C is not smooth but piecewise smooth (consisting of finite number of smooth segments), we can break the integral into pieces
  - If $C = C_1 \cup C_2$ then $\int_C f(x, y)\,\mathrm{d}s = \int_{C_1} f(x, y)\,\mathrm{d}s + \int_{C_2} f(x, y)\,\mathrm{d}s$
- Note: $\int_C f(x, y)\,\mathrm{d}s = \int_{-C} f(x, y)\,\mathrm{d}s$, i.e. the direction of the curve doesn't matter for line integration, sine $\mathrm{d}s > 0$ always
  - This is unlike the 1D case

Lecture 15, Oct 7, 2022

Line Integrals of Vector Fields

- A vector field is $\vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$
  - For each (x, y, z) in space the vector field associates it with a vector
  - We can also express this as $\vec{F}(x, y, z) = \vec{F}(\vec{r})$
- Once again we assume C is smooth
- Consider the example of computing the work done to a particle in a force field
  - Divide the curve into many segments of Δs_i
  - $W_i = \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*)\Delta s_i$
  - $\vec{T}$ is the unit tangent vector; taking the dot product gets us the amount of force in the direction of

movement

  - $W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,\mathrm{d}s$

- Parametrize $C : \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, t \in [a,b]$, then the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$
- $W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,\mathrm{d}s = \int_a^b \vec{F}(x(t),y(t),z(t)) \cdot \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}\|\vec{r}'(t)\|\,\mathrm{d}t = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t$

Important

To compute the line integral of a vector field $\vec{F}(\vec{r})$ along C:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t$$

where $\mathrm{d}\vec{r} = \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \vec{T}\,\mathrm{d}s$

- Alternatively:
$$\begin{aligned}\int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t \\ &= \int_a^b \left(P\hat{\imath} + Q\hat{\jmath} + R\hat{k}\right) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k}\right)\,\mathrm{d}t \\ &= \int_C P\,\mathrm{d}x + \int_C Q\,\mathrm{d}y + \int_C R\,\mathrm{d}z \\ &= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z\end{aligned}$$
- Note the result of a line integral through a vector field is a scalar

Important

Unlike a line integral through a scalar field, $\int_C \vec{F} \cdot \mathrm{d}\vec{r} = -\int_C \vec{F} \cdot \mathrm{d}\vec{r}$, i.e. the direction matters

Lecture 16, Oct 13, 2022

Fundamental Theorem for Line Integrals

- Recall that the fundamental theorem for single integrals is $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$

Definition

A vector field $\vec{F}$ is conservative if $\vec{F} = \vec{\nabla} f$, i.e. it is the gradient of some scalar function f

The scalar function f is known as the potential function of $\vec{F}$

- Suppose $\vec{F}$ is conservative and let C be a smooth curve given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, t \in [a,b]$
- $\int_C \vec{F}(x,y,z)\,\mathrm{d}\vec{r} = \int_C \vec{\nabla} f(x,y,z)\,\mathrm{d}\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t$
  - Notice $f(\vec{r}(t)) \cdot \vec{r}'(t) = \left(\dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{\jmath} + \dfrac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \dfrac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \dfrac{\mathrm{d}z}{\mathrm{d}t}\hat{k}\right)$

  - From the chain rule, this is equal to $\dfrac{\mathrm{d}f}{\mathrm{d}t}$
  - Therefore $\displaystyle\int_C \vec{F}(x,y,z)\,\mathrm{d}\vec{r} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\vec{r}(t))\,\mathrm{d}t = f(\vec{r}(b)) - f(\vec{r}(a))$

Theorem

Fundamental theorem for line integrals:

$$\int_C \vec{F}\cdot\mathrm{d}\vec{r} = \int_C \vec{\nabla} f\cdot\mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

As a consequence $\vec{F} = \vec{\nabla} f \implies \displaystyle\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \oint_C \vec{\nabla} f\cdot\mathrm{d}\vec{r} = 0$ where C is any closed curve

- For a conservative vector field, line integrals are path independent
- This also works in reverse; if $\displaystyle\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = 0$ for all piecewise smooth closed curves C, then $\vec{F}$ is conservative

Determining Conservativeness

- If a vector field is conservative, then $\vec{F}(x,y)P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath} = \vec{\nabla} f$
- Therefore $P = \dfrac{\partial f}{\partial x}, Q = \dfrac{\partial f}{\partial y}$, so by Clairaut's Theorem $\dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y} \implies \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Important

A two variable function $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

For a three-variable function $\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + S(x,y,z)\hat{k}$, the requirement is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial S}{\partial y}, \frac{\partial P}{\partial z} = \frac{\partial S}{\partial x}$$

- For multiple variables, all mixed partials have to be equal

Lecture 17, Oct 14, 2022

Green's Theorem

- Green's theorem connects a line integral over a closed curve C to a double integral over the region R it encloses
- Simple curve: a curve that does not intersect itself except at the endpoints
- Curves can have orientation; a positive oriented curve has the inside of the curve to the left as you go around it (think unit circle)

Theorem

For a positively oriented, piecewise smooth, simple closed curve C in 2D bounding the region R, if $P(x,y), Q(x,y)$ and their first partials are continuous over R, then

$$\oint P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Green's Theorem: Proof of Special Case

- Consider a region that can be expressed as both type 1 and 2: $R = \begin{cases} \{\,(x,y) \mid a \le x \le b, y_1(x) \le y \le y_2(x)\,\} \\ \{\,(x,y) \mid x_1(y) \le x \le x_2(y), c \le y \le d\,\} \end{cases}$
- For the type 1 form, $\displaystyle\oint P(x,y)\,\mathrm{d}x = \int_{C_1} P(x,y)\,\mathrm{d}x + \int_{C_2} P(x,y)\,\mathrm{d}x$
$$\begin{aligned}
&= \int_a^b P(x,y_1(x))\,\mathrm{d}x - \int_a^b P(x,y_2(x))\,\mathrm{d}x \\
&= -\int_a^b \left(P(x,y_2(x)) - P(x,y_1(x))\right)\mathrm{d}x \\
&= -\int_a^b \left[P(x,y)\right]_{y=y_1(x)}^{y=y_2(x)}\mathrm{d}x \\
&= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\,\mathrm{d}y\,\mathrm{d}x \\
&= -\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}A
\end{aligned}$$
  - C_1 is the bottom path $y = y_1(x)$, C_2 is the top path $y = y_2(x)$
- For the type 2 form, $\displaystyle\oint Q(x,y)\,\mathrm{d}y = \int_{C_3} Q(x,y)\,\mathrm{d}y + \int_{C_4} Q(x,y)\,\mathrm{d}y$
$$\begin{aligned}
&= -\int_c^d Q(x_1(y),y)\,\mathrm{d}y + \int_c^d Q(x_2(y),y)\,\mathrm{d}y \\
&= \int_c^d \left(Q(x_2(y),y) - Q(x_1(y),y)\right)\mathrm{d}y \\
&= \int_c^d \left[Q(x,y)\right]_{x=x_2(y)}^{x=x_1(y)}\mathrm{d}y \\
&= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R \frac{\partial Q}{\partial x}\,\mathrm{d}A
\end{aligned}$$
  - C_3 is the left path $x = x_1(y)$, C_4 is the right path $x = x_2(y)$
- If we combine the two, we get: $\displaystyle\oint_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A$

Lecture 18, Oct 14, 2022

Parametric Surfaces

- We can parametrize a surface as $\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}, (u,v) \in D$
  - We need 2 parameters since a surface is a 2-dimensional construct
- The simplest way of doing so is to find $z = f(x,y)$, then parameterize $x = u, y = v \implies S = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$

- This is not always easy or possible to do

- A surface can have multiple parameterizations
- If we fix one parameter, we get a function of the other parameter which is a 1D curve slice of the surface
  - Let $P = (u_0, v_0)$
  - $\left.\dfrac{\partial \vec{r}(u,v)}{\partial v}\right|_{(u_0,v_0)} = \vec{r}_v(u_0, v_0)$ is a tangent vector in the direction of v at P
  - $\left.\dfrac{\partial \vec{r}(u,v)}{\partial u}\right|_{(u_0,v_0)} = \vec{r}_u(u_0, v_0)$ is a tangent vector in the direction of u at P
  - If we assume $\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0) \neq \vec{0}$ at this point, then this cross product is normal to the surface

Definition

A surface is smooth if for every point on S, $\vec{r}_u \times \vec{r}_v \neq \vec{0}$

- The local tangent plane will be normal to $\vec{r}_u \times \vec{r}_v$ and contains the point P; using this we can obtain a tangent plane

Definition

The surface area of a parametric surface can be found by

$$S = \iint_D \|\vec{r}_u \times \vec{r}_v\| \,\mathrm{d}u \,\mathrm{d}v$$

where $S : \vec{r}(u,v), (u,v) \in D$

- Note that in the special case where $\begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases}$, this just reduces to $\iint_D \sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1} \,\mathrm{d}x \,\mathrm{d}y$, which we have seen before

Surface Integrals

- We can extend integration over planar surfaces to any general surface with a surface integral

Definition

The surface integral of a continuous scalar function $f(x,y,z)$ over a smooth, parametrized surface $S : \vec{r}(u,v), (u,v) \in D$ is given by

$$\iint_S f(x,y,z) \,\mathrm{d}S = \iint_D f(x(u,v), y(u,v), z(u,v)) \|\vec{r}_u \times \vec{r}_v\| \,\mathrm{d}u \,\mathrm{d}v$$

- We can also sum together multiple piecewise smooth surfaces

Lecture 19, Oct 20, 2022

Surface Integrals of Vector Fields

- Orientation starts to matter when we talk about vector functions!
  - Some surfaces are orientable, that is, two sided
  - Some surfaces are non-orientable, e.g. a Möbius strip

- A surface has two normal vectors; for an orientable surface, we can define a positive side and a negative side
  - For an open surface, the sign convention is arbitrary unless specified
  - For a closed surface, the convention is always outside is positive
- Consider $S : \vec{r}(u, v)$, a normal to the surface is $\vec{N} = \vec{r}_u \times \vec{r}_v$; the unit normal is $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$
  - If the surface is $z = f(x, y)$, then $\vec{n} = \dfrac{-\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$
- Consider some imaginary, smooth, orientable surface and some fluid with density $\rho(x, y, z)$ and velocity $\vec{V}(x, y, z)$; we can use a surface integral over the vector field to find out how much mass passes through this surface per unit time (the flux)
  - Divide the surface into subregions; for a subregion S_i, the volume of fluid passing through per unit time is $\vec{V} \cdot \vec{n}$
  - Therefore the mass flow through the surface is $\iint_S \rho(x, y, z)\vec{V}(x, y, z) \cdot \vec{n}\,\mathrm{d}S = \iint_S \vec{F} \cdot \vec{n}\,\mathrm{d}S = \iint_S \vec{F} \cdot \mathrm{d}\vec{S}$
  - This is known as *flux integration*
  - Note $\mathrm{d}S = \|\vec{r}_u \times \vec{r}_v\|\,\mathrm{d}u\,\mathrm{d}v$ and $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, so $\vec{n}\,\mathrm{d}S = \mathrm{d}\vec{S} = \vec{r}_u \times \vec{r}_v$

Definition

The flux of the vector field $\vec{F}(x, y, z)$ over a surface $S : \vec{r}(u, v)$ is given by

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S \vec{F} \cdot \vec{n}\,\mathrm{d}S = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\,\mathrm{d}u\,\mathrm{d}v$$

- For a closed surface, the flux is positive if there is a net outflow through the surface

Lecture 20, Oct 21, 2022

Divergence

Definition

Let $\vec{F} = \vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$ be a differentiable vector field, then the divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(P\hat{i} + Q\hat{j} + R\hat{k}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- The divergence is a scalar field
- $\vec{\nabla} \cdot \vec{F}$ is a measure of the local "outgoingness" of the vector field at each point
  - If $\vec{\nabla} \cdot \vec{F} < 0$, the point is a sink – vectors around it go towards it
    - If $\vec{F}$ is a velocity field, with time particles accumulate at this point, increasing the density with time
  - If $\vec{\nabla} \cdot \vec{F} > 0$, then point is a source – vectors around it move away from i
    - Likewise particles move away from it and decrease the density with time
  - If $\vec{\nabla} \cdot \vec{F} = 0$, the density of the fluid around that point does not change
    - In an incompressible fluid, this is one of the governing equations

Curl

Definition

Let $\vec{F} = \vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ be a differentiable vector field, then the curl of $\vec{F}$ is

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

- Unlike divergence, curl is a vector field
- The curl vector is associated with the local rotation of the vector field
  - The direction of the vector is the axis of rotation, the magnitude of the vector is the velocity of rotation
- $\vec{\nabla} \times \vec{V}$ where $\vec{V}$ is a velocity field is the vorticity

Properties of Divergence and Curl

- Let $f = f(x,y,z)$ be a scalar function and $\vec{F} = \vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ be a vector field, both with continuous first partials, then:
  1. $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ (curl of a gradient is zero)
  2. If $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = 0$
     - This essentially comes from the first property; if $\vec{F}$ is conservative then it's a gradient of some scalar function, and we know the curl of a gradient is zero
  3. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$, (divergence of curl is zero)
- $\operatorname{div}(\vec{\nabla} \cdot f) = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla}^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ is known as the Laplace operator

Stokes' Theorem

- Stokes' theorem is the 3D extension of Green's theorem
  - Green's theorem connects the line integral around a closed *planar* curve C and the region R inside
  - Stokes' theorem does this in 3 dimensions, where the curve and its enclosed region is not necessarily planar
    * Think about a bubble surface made by a wire loop

Theorem

Let S be an orientable, piecewise smooth surface that is bound by a simple, closed, piecewise smooth and positively oriented boundary curve C; if $\vec{F}$ is a vector field with continuous first partials over S, then

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, \mathrm{d}S$$

- Stokes' theorem relates the line integral around the boundary of a surface to the surface integral of the curl over that surface
- The orientations of $\vec{n}$ and C must match the right-hand rule: curl your fingers in the direction of C, and the direction your thumb is pointing is the direction of positive $\vec{n}$
  - If you walk along the curve with your head in the direction of $\vec{n}$, if the surface is on your left, then the orientations match
- Green's theorem can be derived from Stokes' theorem:
  - For a surface S in the x-y plane with boundary curve C, its normal vector will be $\vec{n} = \vec{k}$

  - Therefore $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{k} \, \mathrm{d}S$
  - This works out to be equivalent to Green's theorem
- Note the surface of a boundary curve is not unique; we can stretch the surface however we want, as long as the boundary curve stays the same, the surface integral of the curl stays the same

Divergence Theorem

Theorem

Let E be a solid region bounded by the closed, positively (outward) oriented surface S; let $\vec{F}$ be a vector field with continuous first partials in E, then

$$\iint_S \vec{F} \cdot \vec{n} \, \mathrm{d}S = \iiint_E (\vec{\nabla} \cdot \vec{F}) \, \mathrm{d}V$$

- The flux across the boundary of a solid region is equal to the divergence inside the region

Lecture 21, Oct 21, 2022

What is a Fluid?

- With a solid, when a small shear force is applied, we get a small angular deformation, but it stops at equilibrium even when the force is applied continuously
- However the same applied to a fluid will cause the fluid to deform continuously as long as the force is applied
- No slip condition: fluid in direct contact with a solid boundary sticks to it, i.e. it has zero velocity relative to the solid boundary

Definition

A fluid is a substance that deforms *continuously* under the application of a tangential (shear) force, no matter how small the force

- Note, this definition does not address the rate of deformation
- Thus a fluid at rest is always at a state of zero shear force

Studying Fluids

- Two approaches:
  1. Statistical approach: account for the molecular nature of the fluid
     - Macroscopic behaviour of the fluid is determined through statistics and kinetic theory
     - However not practical in most engineering applications
  2. Continuum approach: ignore individual molecules and treat the fluid as a continuous matter
     - More practical for typical engineering applications
     - Will be used in this course
- The continuum approach requires that the macroscopic length scale (size of the system) is much larger than the microscopic length scale (gap between molecules)
  - This is captured in the Knudsen number Kn which is the ratio of the microscopic to the macroscopic length scales
- The continuum approach fails when:
  - Flow through tiny passages, e.g. blood in a vessel
  - Granular flows, e.g. sand
  - Flows with shockwaves, e.g. wake of a supersonic bullet

    - When there is a sudden jump in pressure/temperature, e.g. when a spacecraft enters the atmosphere
- With the continuum approach, we can apply limit concepts from calculus (e.g. work with differentials and infinitesimals) and talk about "points" in the fluid

Important

The continuum approach is only valid when the macroscopic length scale (size of the system) is much larger than the microscopic length scale (gap between molecules) of the fluid

Lecture 22, Oct 27, 2022

Forces on a Fluid

- Two categories:
    - Body forces: developed without physical contact, proportional to the fluid's mass
        * e.g. gravity
    - Surface forces: developed with physical contact at the surface of a fluid element
        * These don't have to be real surfaces
        * Can be broken down into components tangential (shear) or normal to the surface
        * Surface forces result in stresses

Definition

Normal stress is defined as

$$\sigma = \lim_{\delta A \to 0} \frac{\delta F_n}{A}$$

Shear stress is defined as

$$\tau = \lim_{\delta A \to 0} \frac{\delta F_t}{\delta A}$$

- Stress: Force per unit area
    - Stress at a point is defined as the limit as the area decreases to 0
    - Normal stress is defined as $\sigma = \lim_{\delta A \to 0} \frac{\delta F_n}{A}$
    - Shear stress is defined as $\tau = \lim_{\delta A \to 0} \frac{\delta F_t}{\delta A}$
    - Since there are multiple surfaces that can pass through a point, the stress at a point is described completely by specifying 3 stresses on mutually perpendicular planes through the point
- Double subscript notation for stress: τ_{xy} is a stress on the x plane (unit vector in the x direction) acting in the y direction
    - One normal stress σ_{xx}, two tangential stresses τ_{xy}, τ_{xz}, for every surface (in this case x)
    - This means there are 9 such stresses for every point! From these we can form a stress tensor to describe all stress components:
    - Stress tensor: $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$

(Dynamic) Viscosity

- Recall shear stress in a solid is proportional to deformation angle $\delta\alpha$
- Since a fluid never stops deforming, shear stress is instead proportional to the rate of change in the deformation angle
    - $\tau \propto \frac{\mathrm{d}\alpha}{\mathrm{d}t}$

- In a parallel flow field with a velocity gradient (e.g. a boundary layer), the fluid undergoes shear forces as it deforms due to difference in velocity between layers
  - $\mathrm{d}\alpha = \frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}t \implies \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}$
  - The rate of angular deformation of the fluid region is equal to the velocity gradient of the field
- Since $\tau \propto \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}$ for Newtonian fluids we can let $\tau = \mu\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \mu\frac{\mathrm{d}u}{\mathrm{d}y}$, where the proportionality constant μ is the *viscosity*

Definition

The viscosity of a Newtonian fluid is μ defined such that

$$\tau = \mu\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \mu\frac{\mathrm{d}u}{\mathrm{d}y}$$

where $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ is the rate of angular deformation of the fluid element, equal to $\frac{\mathrm{d}u}{\mathrm{d}y}$, the velocity gradient in the fluid

- Newtonian fluids' viscosity are independent of the rate of deformation
  - On a graph of shear stress vs. velocity gradient, a Newtonian fluid starts at the origin and has constant slope
  - For non-Newtonian fluids, this graph is nonlinear and μ can vary with rate of deformation; some may even not start at the origin
    - Shear thinning: with higher rates of deformation, viscosity decreases (e.g. some paints, blood, cookie dough)
    - Shear thickening: with higher rates of deformation, viscosity increases (e.g. cornstarch + water)
    - Bingham plastic: acts like a solid until a certain shear stress, until a certain threshold after which it acts like a fluid (e.g. toothpaste)
      - Graph starts above zero
  - The *local viscosity/apparent viscosity* μ_{ap} is μ at the local conditions
- Viscosity has a strong dependence of temperature:
  - For liquids, viscosity *decreases* with temperature
    - Viscosity is caused by intermolecular forces; at higher temperatures molecules overcome these forces
  - For gases, viscosity *increases* with temperature
    - Viscosity is caused by molecular collisions; at higher temperatures molecules collide more
- "Viscosity" commonly refers to dynamic viscosity as opposed to kinematic viscosity

Definition

The kinematic viscosity $v = \frac{\mu}{\rho}$, where μ is the dynamic viscosity and ρ is the density

Compressibility

Definition

The bulk modulus is defined as $E_V = -\frac{1}{\forall}\frac{\mathrm{d}P}{\mathrm{d}\forall}$ where $\forall$ is the volume

- Bulk modulus measures compressibility; the larger it is, the less compressible the fluid
  - Note an increase in P causes a decrease in $\forall$, which is why there is a negative sign

  - Alternatively can be expressed as $E_V = \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho}$ (note there is no minus sign)
    * $m = \rho\cancel{V} \implies \mathrm{d}m = \rho\mathrm{d}\cancel{V} + \cancel{V}\mathrm{d}\rho = 0 \implies \frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}\cancel{V}}{\cancel{V}}$
  - A truly incompressible flow has $E_V \to \infty$
    * For water it is 2.2×10^9 Pa, so it can be approximated as incompressible
    * Even though air is a lot more compressible, we can still assume it to be incompressible in a low speed flow

Lecture 23, Oct 28, 2022

Hydrostatics

- Hydrostatics deal with fluids at rest or in rigid body motion (e.g. holding a cup and moving)
  - The fluid particles do not move relative to each other
- Recall there are 2 categories of forces on a fluid, body forces and surface forces
  - Surface forces can be split into normal forces (pressure) and shear forces
  - In hydrostatics, since fluid particles don't move relative to each other, there are no shear forces

Pressure

- A normal surface force acting on fluids
- The result of molecular collisions on a real or imaginary surface
  - Molecules colliding with the surface undergo a momentum change, which acts on the surface with a force
- A flat surface may be rough at a microscopic scale, but statistically any tangential forces generated will be cancelled out, so total pressure is normal to the surface
- Pressure is the inward acting normal force per unit area (i.e. normal stress)
- $\mathrm{d}\vec{F} = -p\,\mathrm{d}A\vec{n}$
  - The force vector is proportional to area, and in the opposite direction as the outward facing normal vector
  - We can get a large p with a small F if A is small (e.g. ice skating melting ice)

Pressure at a Point

- Consider a point in a static fluid and an imaginary plane that goes through the point; we can take the limit as the area around the point goes to 0 to define the pressure stress
- There are however infinitely many planes that go through this point, but we can prove that the pressure is equal no matter what direction

Important

Pascal's Law: Pressure at a point in hydrostatic fluid is the same in all directions

- Proof:
  - Consider a container with fluid undergoing rigid body motion
  - Consider a triangular wedge within the container; we have body forces (gravity) and surface forces (pressure) acting on it
    * This includes a slanted plane with some angle θ, sides $\delta x, \delta y, \delta z$ and diagonal δs
  - Pressure forces on the region:
    * On the bottom: $p_z \delta y \delta x$
    * Similarly on the left: $p_y \delta y \delta x$
    * On the slanted surface: $p_s \delta s \delta x$
    * Note we can ignore the pressure forces on the front and back for now

    * Gravitational force $mg = \rho g \left(\frac{\delta x \delta y \delta z}{2}\right)$
  - Force balance: $\begin{cases} \sum F_y = ma_y = p_y \delta z \delta x - p_s \delta s \delta x \sin\theta = \rho \frac{\delta x \delta y \delta z}{2} a_y \\ \sum F_z = ma_z = p_z \delta y \delta x - p_s \delta s \delta x \cos\theta - \rho g \left(\frac{\delta x \delta y \delta z}{2}\right) = \rho \frac{\delta x \delta y \delta z}{2} a_z \end{cases}$
    * Note $\delta s = \frac{\delta y}{\cos\theta} = \frac{\delta z}{\sin\theta}$
    * Substitute and cancel: $\begin{cases} p_y - p_s = \rho \frac{\delta y}{2} a_y \\ p_z - p_s = \rho \frac{\delta z}{2}(a_z + g) \end{cases}$
  - Now take the limit as the size of the region approaches 0 without changing its shape:
    * Let $\delta y \to 0$ then from the first equation we have $p_y = p_s$
    * Let $\delta z \to 0$ then from the second equation we have $p_z = p_s$
    * Therefore $p_s = p_y = p_z$
  - Since these directions are arbitrary, we know pressure is the same for any direction

Basic Pressure Field Equation

- This time, consider a small cubic region inside the fluid with sides $\delta x, \delta y, \delta z$, with center (x_0, y_0, z_0)
- Note: a free surface is denoted in fluid mechanics diagrams with a floating triangular thing
- We want a pressure function $p = p(x, y, z)$
- We again have gravitational and pressure forces
  - We can use a linear approximation to find the pressure at the edge of the region from the pressure in the center
    * Pressure on the left: $p_L = p - \frac{\partial p}{\partial y}\frac{\delta y}{2}$
    * Pressure on the right: $p_L = p + \frac{\partial p}{\partial y}\frac{\delta y}{2}$
    * Similarly for the 4 other faces
  - Now find the pressure forces:
    * Pressure force on the left would be $p_L \delta x \delta z = \left(p - \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z$
    * Pressure force on the right: $\left(p_L = p + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z$
    * Repeat for the other 4 faces
  - Gravitational force: $\rho g \delta x \delta y \delta z$
  - $\sum \delta F_{s,y} = \left(p - \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z - \left(p_L = p + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z$
    * $\sum \delta \vec{F}_s = -\left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}\right)\delta x \delta y \delta z$
  - Total surface force per unit volume is then $-\vec{\nabla} p$
  - Total body force per unit volume is $-\rho g \hat{k}$
- Apply Newton's Second Law:
  - $\sum \delta \vec{F} = \delta m \vec{a}$
  - $-\vec{\nabla} p \delta x \delta y \delta z - \rho g \delta x \delta y \delta z \hat{k} = (\rho \delta x \delta y \delta z)\vec{a}$
  - $-\delta p - \rho g \hat{k} = \rho \vec{a}$
  - Net pressure force per unit volume plus net body force per unit volume is equal to mass times acceleration per unit volume

Equation

General equation of motion for fluids in which there are no shear forces:

$$-\delta p - \rho g \hat{k} = \rho \vec{a}$$

or

$$-\vec{\nabla} p + \rho \vec{g} = \rho \vec{a}$$

if $\vec{g}$ is not in the negative $\hat{k}$ direction

Lecture 24, Oct 28, 2022

Pressure Variations in a Fluid at Rest

- $\vec{a} = 0 \implies \vec{\nabla} p = -\rho g \vec{k} \implies \left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k} \right) = -\rho g \vec{k}$
  - This means $\begin{cases} \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial y} = 0 \\ \frac{\partial p}{\partial z} = -\rho g \end{cases}$
- This means in a fluid at rest $p = p(z)$
  - Pressure is constant along the x and y directions
  - Note the minus sign; as z decreases the pressure goes up
- We can now integrate this: $\Delta p = \int_{z_1}^{z_2} \rho g \, \mathrm{d}z = g \int_{z_1}^{z_2} \rho \, \mathrm{d}z$
  - Case 1: incompressible fluid, ρ constant
    * $\Delta p = -\rho g(z_2 - z_1)$
    * We can write this as $p_2 = p_1 - \rho g h$ or $p_1 = p_2 + \rho g h$ where $h = z_2 - z_1$
      - Note if we're moving down in the fluid, $h < 0$
    * In this case pressure varies linearly with z
  - Case 2: compressible fluid, $\rho = \rho(z)$
    * Define the *specific weight* γ as the weight per unit volume
    * $\gamma = \frac{W}{V} = \frac{mg}{V} = \frac{\rho V g}{V} = \rho g$
    * Therefore the change in pressure is found by integrating the specific weight
    * Note $\rho_{\text{gas}} \ll \rho_{\text{liquid}}$ so by extension, $\left|\frac{\partial p}{\partial z}_{\text{gas}}\right| \ll \left|\frac{\partial p}{\partial z}_{\text{liquid}}\right|$
      - Pressure changes are much smaller in a gas
    * For small distances (e.g. within a tube or a building) in a gas, we can assume pressure to be constant

Summary

For a fluid with no shear force:

$$-\vec{\nabla}p + \rho\vec{g} = \rho\vec{a}$$

If gravity is $\vec{g} = -g\hat{k}$:

$$-\vec{\nabla}p - \rho g\vec{k} = \rho\vec{a}$$

If the fluid is at rest:

$$\Delta p = \int_{z_1}^{z_2} \rho g \, \mathrm{d}z = g\int_{z_1}^{z_2} \rho \, \mathrm{d}z$$

If the fluid is also incompressible:

$$p_2 = p_1 - \rho g h$$

- Example: find the pressure elevation relationship for a perfect ideal gas
  - $p = \rho RT$
  - $\frac{\partial p}{\partial z} = -\rho g \implies \frac{\partial p}{\partial z} = -\frac{p}{RT}g$
  - $\int_{p_1}^{p_2} \frac{1}{p} \, \mathrm{d}p = -\int_{z_1}^{z_2} \frac{g}{RT} \, \mathrm{d}z \implies \ln\frac{p_2}{p_1} = -\frac{g}{RT}\Delta z$
  - $p_2 = p_1 e^{-\frac{g}{RT}(z_2 - z_1)}$

Lecture 25, Nov 3, 2022

Measuring Pressure

- Pressure values must be states with respect to a reference
  - For gauge pressure, this is the local atmospheric pressure or the reference pressure for the gauge
  - For absolute pressure, this is with respect to the absolute zero pressure reference, with is a vacuum
  - Absolute pressure is the sum of gauge and atmospheric pressures
  - Gauge pressure can be positive or negative, but absolute pressure can never be negative
- Manometers can be used to measure pressure
  - Mercury barometers are usually used to measure atmospheric pressure
    - Pressure at the mercury level in the tube $P_A = P_{atm} - \rho_{\mathrm{Hg}} g h = P_{vapour} \approx 0$
  - Piezometers are vertical tubes open at the top; use the level of the liquid to measure the pressure of the liquid in the container
    - $P_A - \rho g h_1 = P_{atm}$
    - Pressure in the container must be greater than that of the atmosphere
    - The fluid inside must be liquid, and the pressure measured must be small
- A U-tube manometer can be used to measure pressures in a gas; a gauge liquid is used

Hydrostatic Forces on Submerged Surfaces

- We know:
  1. The force of pressure is always normal to a surface
  2. No shear stresses
  3. For an incompressible fluid at rest, the pressure varies linearly with depth
- If we want to find the total pressure force on a *planar* surface:
  1. Using integration, we can use a double integral
     - $\vec{F}_p = \iint_A \mathrm{d}\vec{F}_p = \iint_A -p\vec{n} \, \mathrm{d}A$
  2. Using moment of inertia (will not be using in this course)
  3. "Pressure Prism" concept allows us to skip integration using geometry
     - The pressure force can be found by finding the volume of the "pressure prism"

    * One side is the pressure, the other side is the area
- This force acts at the centroid of the prism (not the centroid of the object it acts on!)
- This is the easiest when we have a vertical surface, in which case we get a triangular prism; if the surface does not extend up to the fluid surface, we have a trapezoid
    * The two sides of the triangle are $\rho g h_1$ and $\rho g h_2$ if we consider gauge pressure
    * For the trapezoid, the area is easy but the centroid is hard, so we can break it up into a triangular and rectangular prism, both of which we know the centroids of, and analyze the pressure as 2 forces
- When the surface is not vertical, make sure the pressure forces are normal to the surface!

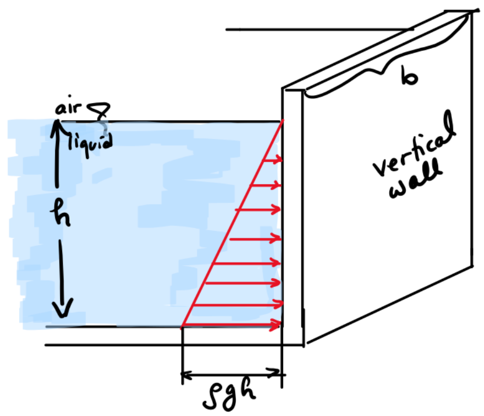

Figure 1: Triangular pressure prism

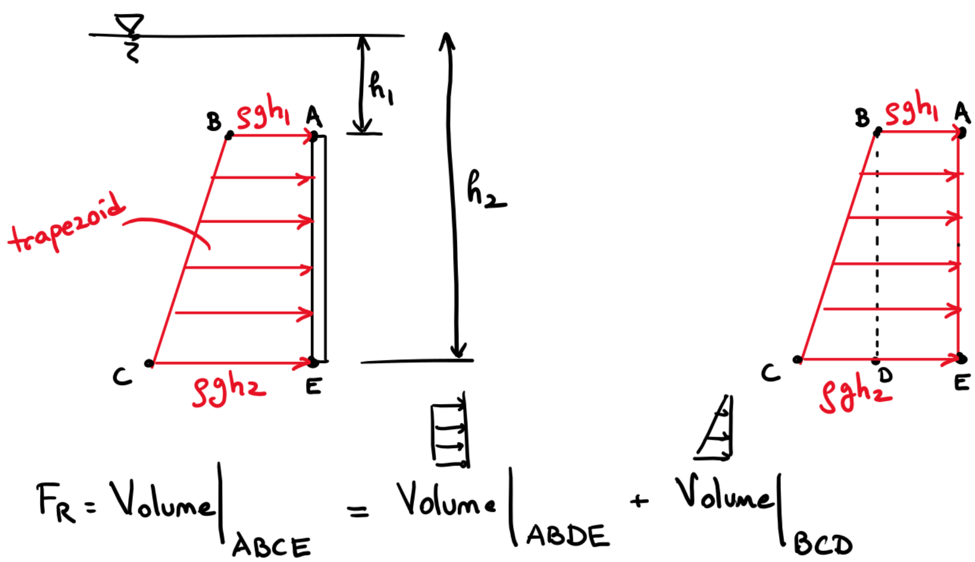

Figure 2: Trapezoidal pressure prism

Lecture 26/27, Nov 4, 2022

Hydrostatic Forces on Curved Surfaces, Integration Method

1. Integration
    - $\vec{F}_p = \iint_A -p\vec{n}\,\mathrm{d}S$

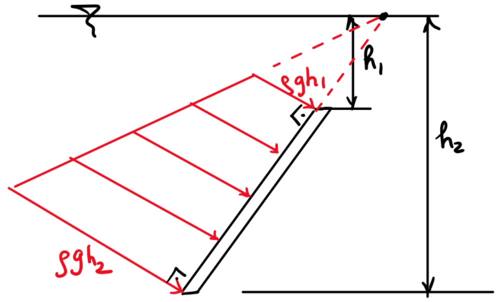

Figure 3: Non-vertical pressure prism

- 3 methods:
  1. Do it in 2 dimensions and parametrize the curve
  2. Do it in 3 dimensions, parametrize the surface
  3. Do it in 3 dimensions, parametrize the surface and use the moment formula $\vec{M}_0 = \vec{r} \times \vec{F} = -p(\vec{r} \times \vec{n})$

2. Graphical method
   - Considers the equilibrium of the fluid volume enclosed by the curved surface of interest and the horizontal and vertical projections of the surface passing through the 2 ends of the curved surface
   - Consider the forces acting on the fluid region, we can draw a free body diagram
   - There are pressure forces on the 3 sides, the curved surface, the horizontal projection, and the vertical projection, as well as the weight of the fluid inside
     - The pressure force on the horizontal projection is constant
     - The pressure force on the vertical projection is similar to a vertical pressure prism
     - The weight is a single force due to gravity
   - Because the fluid is at rest or rigid body motion the forces on this fluid region are in equilibrium, so we can find the force on the planar surface by the sum of the pressure forces on the other 2 sides
   - The force on the curved region is at angle $\alpha = \tan^{-1} \dfrac{F_{RV}}{F_{RH}}$ with magnitude $F_R = \sqrt{F_{RH}^2 + F_{RV}^2}$
     - $F_RH = F_x$, $F_{RV} = F_y + w$ because we need to account for weight
   - The location of application can be found by taking the moment about a convenient axis on the surface over which it acts
     - Note for a curved surface *above* a liquid, weight acts in the opposite direction compared to pressure
   - If the surface is circular, then the resulting pressure force will pass through the centre of the circle
     - Since pressure force is always normal to the surface, all the individual pressure forces will pass through the centre, so the overall force must also pass through the centre

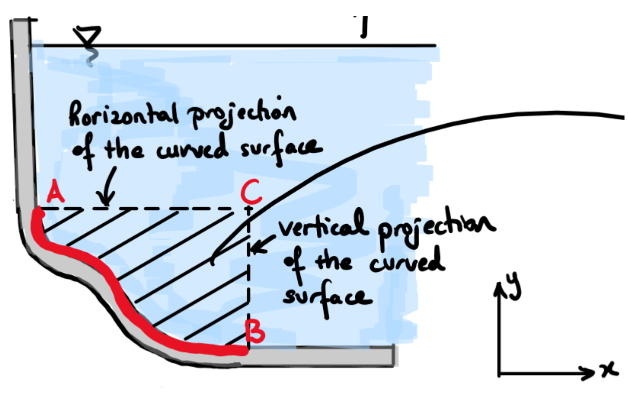

Figure 4: Graphical method

Lecture 28, Nov 17, 2022

Buoyancy and Archimedes' Principle

- Buoyant force is caused by the increase of pressure with depth
- Parts of an object that are submerged deeper experience higher pressure, which gives a net upward force

Theorem

Archimedes' Principle: The buoyant force acting on a submerged body is equal to the weight of the fluid displaced by the solid body, acting in the direction opposite to gravity:

$$\vec{F}_B = \rho_{\text{fluid}} g V \vec{k}$$

- Note that the buoyant force is independent of the density of the object or how deep it is
- For any solid submerged object, consider a region of fluid in the same shape; the region of fluid is in static equilibrium, so its buoyant force should be equal to its weight
  - The buoyant force of the fluid is the same as that of the solid body since they have the same volume
  - Therefore the buoyant force on the object is the same as the weight of the fluid displaced
- For a floating body, the buoyant force caused by the submerged region is in balance with the weight of the body
- If $\rho_b < \rho_f$, the body floats because a partially submerged body displaces more weight of the fluid than the weight of the body
- If $\rho_b = \rho_f$, the body is neutrally buoyant and will be suspended in the fluid
- If $\rho_b > \rho_f$, the body sinks because the buoyant force never matches the weight of the body, even when the entire body is submerged
- Note the buoyant force is proportional to the fluid density, so buoyant forces in gases are usually negligible

Lecture 29, Nov 18, 2022

Boat and Stone Problem

- If you're on a boat with a stone and throw the stone into the water, what happens to the water level?
- When the stone is on the boat, some portion of the buoyant force goes to lift the stone
- The displaced water volume by the stone on the boat is larger than the volume of the stone itself in order to generate the necessary buoyant force, since the stone is denser than the water
- When the stone is thrown into the water, it sinks and displaces its volume to the water; however the boat rises up by an amount that is more than the volume of the stone
- Therefore throwing the stone into the lake decreases the lake level

Stability of Immersed and Floating Bodies

- For immersed or floating bodies, there are two forces, $\vec{F}_B$ which acts through the centroid of the displaced volume and $\vec{W}$ which acts through the center of gravity of the object
- For a completely submerged body, if the body is bottom-heavy then it is stable; if it's top-heavy then it's unstable
  - e.g. in a hot air balloon or a submarine the mass is concentrated at the bottom so they are stable
- For floating bodies, stability depends on the shape since the centroid of the submerged region changes as the body rotates
  - A wide body is stable while a narrow body is not

Hydrostatics Equation Derivation From Integrals

- Consider a fluid volume $\mathcal{V}$ with arbitrary shape, with gravity $\vec{g}$ not necessarily aligned with the axes
- The only assumption is we're dealing with hydrostatic so no shear is involved
- Consider a differential volume element $\mathrm{d}\mathcal{V}$ with mass $\mathrm{d}m = \rho \mathrm{d}\mathcal{V}$
  - The body forces are $\delta \vec{F}_{body} = \rho \vec{g} \,\mathrm{d}\mathcal{V} \implies \sum \vec{F}_{body} = \iiint_{\mathcal{V}} \rho \vec{g} \,\mathrm{d}\mathcal{V}$
  - The surfaces forces are $\delta \vec{F}_{surface} = -p\vec{n} \,\mathrm{d}S \implies \sum \vec{F}_{surface} = -\iint_s p\vec{n} \,\mathrm{d}S$
  - Therefore $\iiint_{\mathcal{V}} \rho \vec{g} \,\mathrm{d}\mathcal{V} - \iint_s p\vec{n} \,\mathrm{d}S = m\vec{a} = \iiint \rho \vec{a} \,\mathrm{d}\mathcal{V}$
  - Using the gradient theorem $\iint_S p\vec{n} \,\mathrm{d}S = \iiint_{\mathcal{V}} \vec{\nabla} p \,\mathrm{d}\mathcal{V}$
    * $\iint_S f\vec{n} \,\mathrm{d}S = \iiint_{\mathcal{V}} \vec{\nabla} f \,\mathrm{d}\mathcal{V}$ where f is a scalar and $\vec{\nabla} f$ is the gradient of f
    * The gradient theorem is the analogue of the divergence theorem
  - We can now combine the integrals as $\iiint_{\mathcal{V}} (\rho \vec{g} - \vec{\nabla} p) \,\mathrm{d}\mathcal{V} = \iiint_{\mathcal{V}} \rho \vec{a} \,\mathrm{d}\mathcal{V}$
  - Since the region is arbitrary we can shrink it arbitrarily; therefore the integrands have to be equal
  - Therefore $\rho \vec{g} - \vec{\nabla} p = \rho \vec{a}$, which is the hydrostatic equation

Fluids in Linear Rigid-Body Motion

- Consider a container partially filled with incompressible liquid moving in a straight path with $\vec{a} = \text{const}$
- What is the shape of the free surface $z_s(y)$?
- From the hydrostatic equation $-\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right) - \rho g \vec{k} = \rho(a_x \hat{i} + a_y \hat{j} + a_z \hat{k})$
  - If the acceleration is in the y and z directions we get $\mathrm{d}p = \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z = -\rho a_y \,\mathrm{d}y - \rho(g + a_z) \,\mathrm{d}z$
  - This gives $p(y, z) = -\rho a_y y - \rho(g + a_z)z + C$
- Boundary condition: At $z = z_s$, $p = p_{atm}$
  - $p_{atm} = -\rho a_y y - \rho(g + az)z_s + C$
  - $z_s = \frac{C - p_{atm}}{\rho(g + az)} - \frac{a_y}{(g + az)} y$
  - Let $C_1 = \frac{C - p_{atm}}{\rho(g + az)}$ then $z_s = C_1 - \frac{a_y}{(g + az)} y$
- The surface of the liquid turns out to be linear, with a slope of $-\frac{a_y}{g + az}$
- At rest the surface should be flat and have the same volume, we can use this to find C_1
  - $\int_0^L w z_s \,\mathrm{d}y = HwL \implies C_1 L - \frac{a_y}{g + a_z}\frac{L^2}{2} = HL \implies C_1 = H + \frac{a_y}{2(g + az)} L$
  - Therefore $z_s(y) = H + \frac{a_y}{g + a_z}\left(\frac{L}{2} - y\right)$
- Alternatively notice along a line of constant pressure $\mathrm{d}p = 0$, if we put this into $\mathrm{d}p = -\rho a_y \,\mathrm{d}y - \rho(g + a_z) \,\mathrm{d}z$ we get $\frac{\mathrm{d}z}{\mathrm{d}y} = -\frac{a_y}{g + a_z}$, since the surface of the fluid is a line of constant pressure
- Notice if $a_y = 0$ the slope is still 0, so the fluid surface stays flat
  - In this case $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ but $\frac{\partial p}{\partial z} = -\rho(g + a_z) \neq 0$
  - If we are e.g. in an elevator, we can integrate $\frac{\partial p}{\partial z}$ and notice a larger pressure gradient in the z direction
- Also note for a fluid in free fall $a_z = -g$ so p is constant throughout the fluid

Summary

The free surface for a fluid undergoing linear rigid-body motion is a flat planar surface given by

$$z_s(y) = H + \frac{a_y}{g + a_z}\left(\frac{L}{2} - y\right)$$

with pressure given by

$$p(y, z) = -\rho a_y y - \rho(g + a_z)z + C$$

Lecture 30, Nov 18, 2022

Fluids in Rotational Rigid-Body Motion

- Consider a cylindrical container with radius R partially filled with liquid, rotated about its centre axis at a constant velocity ω
- Assume no shear, what is the free surface $z = z_s$?
- Using a cylindrical coordinate system (r, θ, z), we have $\begin{cases} a_r = -r\omega \\ a_\theta = 0 \\ a_z = 0 \end{cases}$
  - a_r comes from centripetal acceleration
- Using the hydrostatic equation again, $-\left(\frac{\partial p}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\vec{e}_\theta + \frac{\partial p}{\partial z}\vec{e}_z\right) - \rho g\vec{e}_z = \rho(-r\omega^2)\vec{e}_r$
  - Notice the gradient in cylindrical coordinates has a $\frac{1}{r}$ in the θ term
- Matching the terms: $\begin{cases} -\frac{\partial p}{\partial r} = -\rho r\omega^2 \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \\ -\frac{\partial p}{\partial z} = \rho g \end{cases} \implies \mathrm{d}p = \frac{\partial p}{\partial r}\,\mathrm{d}r + \frac{\partial p}{\partial z}\,\mathrm{d}z = \rho r\omega^2\,\mathrm{d}r - \rho g\,\mathrm{d}z$
- Integrating this we have $p = \frac{\rho r^2\omega^2}{2} - \rho gz + C$
- Again using the boundary condition that $p = p_{atm}$ at $z = z_s$ we have $p_{atm} = \frac{\rho r^2\omega^2}{2} - \rho g z_s + C \implies z_s = \frac{\omega^2}{2g}r^2 + \frac{C - p_{atm}}{\rho g} = \frac{\omega^2}{2g}r^2 + C_1$
- Therefore the free surface is a paraboloid
- Again from the fact that at rest the fluid surface is flat and the volume stays the same, integrating the volume of the paraboloid we can solve for $C_1 = H - \frac{\omega^2 R^2}{4g} \implies z_s = \frac{\omega^2}{4g}(2r^2 - R^2) + H$

Summary

The free surface for a fluid undergoing rotational rigid-body motion is a paraboloid given by

$$z_s(r) = \frac{\omega^2}{4g}(2r^2 - R^2) + H$$

with pressure given by

$$p = \frac{\rho r^2\omega^2}{2} - \rho gz + C$$

Fluid in Motion (Overview)

- Lagrangian vs. Eulerian descriptions:
  - The Lagrangian method describes the motion of individual particles over time, modelling using Newton's laws
    * $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + k(t)\hat{k}$
    * To describe the whole fluid we need to do this for a large number of particles
  - The Eulerian method instead considers regions of space and properties such as the overall fluid velocity in a certain region
    * $\vec{V} = \vec{V}(x, y, z, t)$
  - Eulerian method is a lot easier to work with
- 3 concepts are used to visualize a flow:
  - Streamlines: Imaginary lines that are tangent to the local fluid velocity at every point at an instant in time
    * Streamlines can vary through time
    * By definition, fluid cannot ever flow across a streamline
    * Streamlines can be determined using Particle Image Velocimetry (PIV)
    * Streamtubes: A set of streamlines passing through all points of a closed curve
      - Fluid can't cross a streamtube, so these can be used to isolate a region of a flow
    * Stream filament: A streamtube that is thin enough that the cross-sectional velocity is effectively constant
  - Pathlines: The line traced out by a specific fluid particle as it moves in time
    * Lagrangian concept
    * Experimentally, they can be visualized by putting tracer particles in the fluid and taking long-exposure photographs
  - Streaklines: The line formed by connecting all fluid particles that pass through a point
    * Experimentally, they can be visualized by injecting dye or smoke into the fluid at a particular fixed point
- Steady flows are flows in which fluid properties at a point do not change with time
  - Fluid properties are a function of space only, not time
  - In a steady flow, the streamline, streakline, and pathline all coincide
- Unsteady flows are flows in which the properties can be time-dependent
  - In an unsteady flow, the streamline, streakline and pathline are different
  - Two particles that pass through the same point don't necessarily end up in the same place, so the streakline connecting the points won't be the same as the pathline
  - The streamline will vary with time

Lecture 31, Nov 24, 2022

Fluid in Motion (Continued)

- Viscous regions are regions in which the frictional effects are significant
  - In these regions we need to consider shear forces due viscous effects
  - Boundary layers are viscous regions; in these areas the friction slows down the velocity and creates a gradient
  - This also includes wake regions
- Inviscid regions are regions in which the frictional effects are negligible
  - Doesn't mean that viscosity is zero, but we don't have to consider shear stresses
- Flow dimensionality: a flow is n-dimensional if flow properties vary in n dimensions
  - e.g. consider a uniform flow entering a pipe; near the entrance region, the flow is 2 dimensional since the velocity profile changes as you go down the pipe; after the flow is fully developed, the flow becomes 1 dimensional as the velocity profile stops changing
  - This depends on what coordinate system is used! e.g. with the pipe example, the flow is 2 or 1 dimensional in cylindrical coordinates, but is 3 or 2 dimensional in Cartesian coordinates

- Laminar flow: highly ordered fluid motion with smooth layers
- Turbulent flow: Highly disordered fluid motion with a lot of velocity fluctuations
  - Typically occurs at high velocities
- Transitional flow: a flow that alternates between laminar and turbulent

Definition

The Reynolds number is defined as

$$\mathrm{Re} = \frac{\rho V L}{\mu}$$

At low Reynolds numbers, flow tends to be laminar; at high Reynolds numbers, flow tends to be turbulent

- The Reynolds number is a non-dimensional number that characterizes the flow; most fluids have a critical Reynolds number at which they transition from laminar to turbulent
  - A ratio of inertial to viscous forces
  - Higher viscosity makes the flow more laminar
  - Higher velocity, density or characteristic length makes the flow turbulent

Mass Conservation

- For now, assume steady (time-independent), incompressible (constant ρ) and 1 dimensional flow
- The volumetric flow rate $\dot{V\!\!\!/}$ is the rate of fluid passing through a given region per unit time, in units of m^3/s
- The volumetric flow rate through a surface A is $V\!\!\!/ = \iint_A \vec{V} \cdot \vec{n}\,\mathrm{d}A = \iint_A V_n\,\mathrm{d}A$ where V_n is the normal component of velocity
  - In the special case where V_n is constant, $\dot{V\!\!\!/} = V_n A$
  - Multiply by ρ for the mass flow rate $\dot{m}$
- Since the flow is steady, the volume flow rate along a streamtube does not change therefore we have $\dot{m}_1 = \dot{m}_2$
  - Additionally for 1D flow ρ and V are constant over an area, so $\dot{m}_1\dot{m}_2 = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$

Equation

Conservation of mass for a steady 1D flow:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

for a compressible flow, or

$$V_1 A_1 = V_2 A_2$$

for an incompressible flow

- Due to the 1D assumption, velocities would need to be given as average velocities

Energy Conservation (Euler and Bernoulli Equation)

- In fluids there are 3 types of energy: potential, kinetic, and pressure
- Assume steady and incompressible flow in an inviscid region (note inviscid implies laminar flow, since turbulence introduces frictional losses)
  - Friction will lead to energy losses, which is why this only works in inviscid regions
  - Therefore this is only an approximation and will not work in the boundary region (close to solid surface) and wake (turbulent) regions
- Consider a cylindrical differential element along the pathline/streamline

  - Use the coordinate system of s tangent to the streamline, s normal to the streamline
- For the s direction, $\sum F_s = ma_s$
  - $\sum F_s = W_s + \sum F_{p,s}$ (sum of weight and pressure forces)
  - $\sum F_s = -\rho g \,\mathrm{d}A \,\mathrm{d}s \sin\theta + p \,\mathrm{d}A - (p + \mathrm{d}p) \,\mathrm{d}A = -\rho g \,\mathrm{d}A \,\mathrm{d}z - \mathrm{d}p \,\mathrm{d}A$
  - $a_s = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = V\frac{\mathrm{d}V}{\mathrm{d}s}$
  - $ma_s = \rho \,\mathrm{d}A \,\mathrm{d}s \cdot V\frac{\mathrm{d}V}{\mathrm{d}s}$
  - Equating the two, we get Euler's equation: $V \,\mathrm{d}V + g \,\mathrm{d}z + \frac{1}{\rho} \,\mathrm{d}p = 0$

Equation

Euler's equation:

$$V \,\mathrm{d}V + g \,\mathrm{d}z + \frac{1}{\rho} \,\mathrm{d}p = 0$$

valid for steady, inviscid flow along a streamline (can be compressible)

Lecture 32/33, Nov 25, 2022

Bernoulli's Equation

- Under the assumption of incompressible flow (constant ρ) we can integrate Euler's equation and obtain Bernoulli's equation
- $\frac{V^2}{2} + gz + \frac{p}{\rho} = \text{const}$
- Bernoulli's equation is an energy conservation equation; all terms have units of energy per unit mass
  - $\frac{V^2}{2}$ is kinetic energy per unit mass
  - gz is potential energy per unit mass
  - $\frac{p}{\rho}$ is pressure energy per unit mass

Summary

For steady, inviscid flow along a streamline, Euler's equation is given by

$$V \,\mathrm{d}V + g \,\mathrm{d}z + \frac{1}{\rho} \,\mathrm{d}p = 0$$

If the flow is also incompressible, then Bernoulli's equation applies:

$$\frac{V^2}{2} + gz + \frac{p}{\rho} = \text{const}$$

- Note that Euler's/Bernoulli's equation only works along a streamline!
- Additionally since they rely on energy conservation, between the two points there must be no heat loss and no shaft work
- If we apply the same analysis normal to the streamline we get $\frac{\mathrm{d}p}{\rho} + \frac{V^2}{R} \,\mathrm{d}n + g \,\mathrm{d}z = 0$
  - For a straight streamline $R \to \infty$ and we just have $\frac{\mathrm{d}p}{\rho} + g \,\mathrm{d}z = 0$
  - If the flow is incompressible we can integrate this and get $p_1 - p_2 = \rho g(z_2 - z_1)$, which is the hydrostatic equation, but this time for a straight, incompressible flow

Static, Dynamic, Total and Stagnation Pressures

- Multiplying the Bernoulli equation by ρ we get $p + \rho\frac{V^2}{2} + \rho g z = P_T$
  - p is the static pressure, the pressure in the flow that does not incorporate any dynamic effects
  - $\rho\frac{V^2}{2}$ is the dynamic pressure, the pressure rise when the fluid is stopped isentropically
  - $\rho g z$ is the hydrostatic pressure (but not exactly, since it depends on the reference level for z)
  - P_T is the total pressure
  - $p + \rho\frac{V^2}{2}$ is the stagnation pressure, the pressure observed when the fluid is brought to a stop (includes both the static and dynamic pressures)

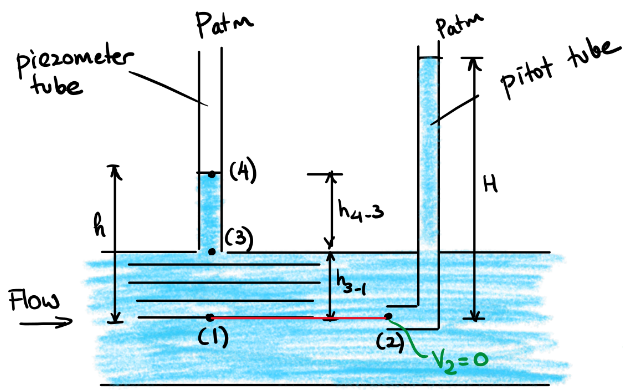

Figure 5: Measurement of static and dynamic pressures

- Static and stagnation pressures can be measured through a piezometer tube and a pitot tube, from which the flow velocity can be calculated

Reynolds Transport Theorem

- Two approaches of examining the flow:
  - Control volume approach (Eulerian): consider a region fixed in space, which fluid can flow in or out of
  - System approach (Lagrangian): consider a fixed collection of fluid particles, which moves with the flow
- Equations used in solid analysis (e.g. Newton's laws) apply to systems, but for fluids it's easier to use control volumes; Reynolds Transport Theorem links the two approaches
- Let B be some mass dependent property and $B = mb$ where b is that property per unit mass
  - $B_{sys} = \int_{sys} b\rho \mathrm{d}V\!\!\!/ \implies \frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{sys} b\rho \mathrm{d}V\!\!\!/$
  - $B_{cv} = \int_{cv} b\rho \mathrm{d}V\!\!\!/ \implies \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{cv} b\rho \mathrm{d}V\!\!\!/$

Equation

Reynolds Transport Theorem:

$$\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} + \oiint_S \rho b \vec{V} \cdot \mathrm{d}\vec{A}$$

where $\mathrm{d}\vec{A} = \vec{n}\mathrm{d}A$ and S is the boundary of the control volume

- In the simplified case of single-inlet, single-outlet, 1D flow along a streamtube, $\frac{\mathrm{d}B_{sys}}{\mathrm{d}t} = \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} + \dot{B}_{out} -$

$\dot{B}_{in} = \frac{\mathrm{d}B_{cv}}{\mathrm{d}t} + \dot{m}_{out}b_{out} - \dot{m}_{in}b_{in}$

- Using this we can derive the most general form of the continuity equation as $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathrm{d}V + \oiint_S \rho\vec{V}\cdot\mathrm{d}\vec{A} = 0$
- Note the selection of control volume can make a problem easier; e.g. in the case of a body moving through a fluid, it's often best to have the control volume move with the body, so flow inside the control volume is steady

Lecture 34, Dec 1, 2022

Momentum Equation

- The Eulerian form of the momentum equation: $\sum \vec{F}_{cv} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{M}_{cv} + \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$
  - For a steady flow, $\sum F_{cv} = \dot{\vec{M}}_{out} - \dot{\vec{M}}_{in}$
- In a 1D problem: $\sum \vec{F}_{cv} = \sum \dot{m}_{out}\vec{V}_{out} - \sum \dot{m}_{in}\vec{V}_{in}$
- We draw force diagrams of forces acting on the control volume similar to a free body diagram for solid analysis
- Example: Flow inside a pipe
  - The forces acting on the control volume would include the shear force on the walls $\tau\pi DL$, the net pressure force $(p_1 - p_2)\pi\frac{D^2}{4}$, and weight $\rho g\pi\frac{D^2}{4}L$
  - However if we make the control volume bigger than the pipe wall, we no longer have shear forces; however we'd get a force due to the rest of the pipe wall, and the weight of the pipe material

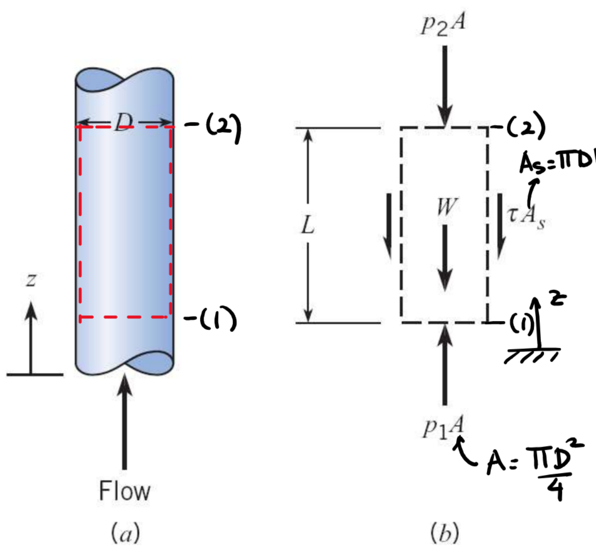

Figure 6: Flow inside a pipe

- Choice of the control volume depends on what information we want to find
- When expressing pressure forces acting on control surfaces, we ignore atmospheric pressure, because it cancels out

Lecture 35, Dec 2, 2022

Models of the Fluid

- To develop any governing equation, we start from physical principles (e.g. conservation of mass, energy), use a suitable model of the fluid, and then derive mathematical equations
  - We have seen the finite control volume and finite system, now we will see the infinitesimal control volume and infinitesimal system
- We take an infinitesimally small control volume and system moving with the flow, such that all properties in this volume are constant

Substantial Derivative

- Consider an infinitesimally small fluid element, at point 1 at $t = t_1$ and travelling to point 2 at $t = t_2$
- Let $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ be the velocity field, where u, v, w are functions of (x, y, z, t)
  - Let $\rho = \rho(x, y, z, t)$ be the density
- Initially the fluid particle has $\rho_1 = \rho(x_1, y_1, z_1, t_1)$ and velocity $\vec{V_1}$; at time $t = t_2$ the particle has $\rho_2 = \rho(x_2, y_2, z_2, t_2)$ and $\vec{V_2}$
- Taylor expand density around point 1: $\rho_2 = \rho_1 + \frac{\partial \rho}{\partial x}(x_2 - x_1) + \frac{\partial \rho}{\partial y}(y_2 - y_1) + \frac{\partial \rho}{\partial z}(z_2 - z_1) + \frac{\partial \rho}{\partial t}(t_2 - t_1) + \cdots$
  - $\frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{\partial \rho}{\partial x}\frac{x_2 - x_1}{t_2 - t_1} + \frac{\partial \rho}{\partial y}\frac{y_2 - y_1}{t_2 - t_1} + \frac{\partial \rho}{\partial z}\frac{z_2 - z_1}{t_2 - t_1} + \frac{\partial \rho}{\partial t}$
  - If we take $\lim_{t_2 \to t_1}$, we get $\frac{\mathrm{D}\rho}{\mathrm{D}t} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}$
    * The notation $\frac{\mathrm{D}\rho}{\mathrm{D}t}$ indicates that we're following the same system
    * This is called a *substantial derivative*
- The substantial derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ is Lagrangian, while the right hand side with all the partial derivatives is Eulerian

Definition

The substantial derivative operator is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$$

where $\vec{V} \cdot \vec{\nabla}$ is the convective derivative

- The substantial derivative is made of the local derivative, time rate of change at a fixed point due to unsteady fluctuations, and the *convective derivative* $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, which is the time rate of change as a result of the movement of fluid
- The substantial derivative is a total derivative with respect to time

Divergence of Velocity

- $\vec{\nabla} \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ can be interpreted as the time rate of change of the volume of an infinitesimal moving system per unit volume
- As the system is moving, its volume is continuously changing
- $\vec{\nabla} \cdot \vec{V} = \frac{1}{\delta\Omega}\frac{\mathrm{D}(\delta\Omega)}{\mathrm{D}t}$
- Note that if we have an incompressible flow, constant density means $\delta\Omega_{sys}$ must be constant, so $\frac{\mathrm{D}(\delta\Omega)}{\mathrm{D}t} = 0$, this means $\vec{\nabla} \cdot \vec{V} = 0$

General Continuity Equation

- We're starting with mass conservation, move through the 4 different models to get the continuity equation
- Recall for the finite control volume we have $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_\Omega \rho \,\mathrm{d}\Omega + \oiint_S \rho\vec{V}\cdot \mathrm{d}\vec{S} = 0$ which we derived from RTT
- For the finite system, we follow the system and take $\frac{\mathrm{D}m_{sys}}{\mathrm{D}t}$, which by mass conservation should be 0, which gives us $\frac{\mathrm{D}}{\mathrm{D}t}\iiint_\Omega \rho\,\mathrm{d}\Omega = 0$
- For the infinitesimal control volume:
  - Along a side we have the flow rate as $\left(\rho u + \frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z$
  - The net mass flow rate out of the control volume is $\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
  - This is equal to $-\frac{\partial}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ due to conservation of mass
  - Combined this gives us $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$
- Now consider the infinitesimal system:
  - $\frac{\mathrm{D}(\delta m)}{\mathrm{D}t} = 0$ where $\delta m = \rho\delta\Omega$
  - $\frac{\mathrm{D}(\delta m)}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta\Omega) = \rho\frac{\mathrm{D}}{\mathrm{D}t}(\Delta\Omega) + \delta\Omega\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$
  - $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\frac{1}{\delta\Omega}\frac{\mathrm{D}(\Delta\Omega)}{\mathrm{D}t}\right) = 0$
  - This gives us $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\vec{\nabla}\cdot\vec{V} = 0$
- This gave us 4 general forms of the continuity equation:
  1. $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_\Omega \rho\,\mathrm{d}\Omega + \oiint_S \rho\vec{V}\cdot\mathrm{d}\vec{S} = 0$
  2. $\frac{\mathrm{D}}{\mathrm{D}t}\iiint_\Omega \rho\,\mathrm{d}\Omega = 0$
  3. $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$
  4. $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\vec{\nabla}\cdot\vec{V} = 0$
- For a steady flow, $\frac{\partial}{\partial t} = 0$ but $\frac{\mathrm{D}}{\mathrm{D}t} \neq 0$, so we have:
  1. $\oiint_S \rho\vec{V}\cdot\,\mathrm{d}\vec{S} = 0$
  2. $\frac{\mathrm{D}}{\mathrm{D}t}\iiint_\Omega \rho\,\mathrm{d}\Omega = 0$
  3. $\vec{\nabla}\cdot(\rho\vec{V}) = 0$
  4. $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\vec{\nabla}\cdot\vec{V} = 0$
- For an compressible flow, ρ is constant and so:
  1. $\oiint_S \rho\vec{V}\cdot\mathrm{d}\vec{S} = 0$
  2. $\frac{\mathrm{D}}{\mathrm{D}t}\Omega = 0$
  3. $\vec{\nabla}\cdot\vec{V} = 0$
     - Note 3 and 4 became the same
- All these forms are mathematically equivalent

Lecture 36, Dec 2, 2022

Open Channel Flows

- Flows of liquid with free surface exposed to atmospheric pressure pressure
  - This free surface introduces an extra degree of freedom
  - This allows aves to form
- Waves move at a speed of c_0, which is not the same as the velocity of the individual fluid particles
- We want a control volume that moves with the wavefront
  - Assuming a wave height much less than the liquid height, $\delta y \ll y$, then $c_0 = \sqrt{gy}$
  - The wave speed depends on the liquid depth; this is why tsunamis form, since waver level is very deep in the ocean
- For open channel flows, we define the Froude number $\mathrm{Fr} = \dfrac{V}{\sqrt{gy}}$, the ratio of the fluid speed divided by the wave speed; the Froude number governs the character of the flow in open channels
  - $\mathrm{Fr} < 1$ is subcritical flow
    - The waves drift due to the velocity, but the wave still moves both down and upstream since the wave is able to travel faster than the fluid
  - $\mathrm{Fr} = 1$ is critical flow
    - The wave velocity matches the fluid velocity, so the wavefront stays in place
  - $\mathrm{Fr} > 1$ is supercritical flow
    - The waves only move downstream since the fluid pushes it down faster than it can go upstream
- Open channel flows are similar to compressible flows, in which the Mach number is used
  - In supercritical flow the wavefront is analogous to the shockwave in supersonic flow

Compressible Flows

- Incompressibility is always only an approximation
  - The constant density assumption greatly simplifies problems
  - This is valid in a slow moving fluid
- In compressible flows we need fluid dynamics and thermodynamics
- A weak pressure wave is defined as a sound wave
  - The pressure wave is travelling at the speed of sound, but not the fluid particles
- Like in the open channel flow we again look at a control volume moving with the wavefront and assume 1D travel
  - Using continuity and momentum we get $c^2 = \left(\dfrac{\partial p}{\partial \rho}\right)_s$
    - Note this is at constant entropy, because the disturbance is very small and we're not adding heat
- For an ideal gas, $\dfrac{P_2}{P_1} = \left(\dfrac{\rho_2}{\rho_1}\right)^\gamma = \left(\dfrac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$
  - $\dfrac{p}{\rho^\gamma}$ is constant
  - $c = \sqrt{\gamma RT}$ for an ideal gas
- More generally for any fluid we can use the bulk modulus and get $c = \sqrt{\dfrac{E_v}{\rho}}$
- In a truly incompressible medium, $E_v \to \infty$ which means $c \to \infty$
- The Mach number is defined as $M = \dfrac{V}{c}$
  - Note this is a variable from point to point
  - We generally use M_∞, the free stream Mach number
- We can categorize the flow based on Mach number:
  - $M_\infty \leq 0.3$ means the flow is incompressible
  - $M_\infty > 0.3$ means the flow is incompressible
  - $0.8 \leq M_\infty \leq 1$ gives transonic flow

- $M_\infty \geq 5$ gives hypersonic flow

Simplified Compressible Flows

- We will assume steady, 1D, isentropic (adiabatic and inviscid), and compressible flow
- $\rho V A$ is constant, so we have $\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}A}{A} + \frac{\mathrm{d}V}{V} = 0$ as an alternative form of the continuity equation
- Pressure work can be derived as $p_1 A_1 V_1 - p_2 A_2 V_2$
- Using RTT on energy balance, we get the compressible Bernoulli equation: $\frac{p}{\rho} + e + \frac{V^2}{2} + gz = \text{const}$ where e is the total internal energy per unit mass
  - In terms of enthalpy $h + \frac{V^2}{2} + gz = \text{const}$
- For high-speed flows, potential energy of the fluid is negligible; if we imagine that we can adiabatically slow the fluid to zero, then we get $h + \frac{v^2}{2} = h_0$, the *stagnation enthalpy* or *total enthalpy*
  - Kinetic energy converts to enthalpy
  - All the kinetic energy goes to an increase in internal energy (temperature) and pressure energy
- We can find properties of the fluid at stagnation:
  - $c_p(T - T_0) + \frac{V^2}{2} = 0 \implies T_0 = T + \frac{V^2}{2c_p}$
  - T is the static temperature, the regular temperature we know
  - $\frac{V^2}{2c_p}$ is the dynamic temperature, the temperature rise in the stagnation process
  - T_0 is the stagnation or total temperature, the temperature we get when we bring the fluid to a stop adiabatically
  - For a very high speed flow we have $T_0 > T$ and kinetic energy is important, but for low speed flows we have $T_0 \approx T$ since kinetic energy is negligible
- We can get properties such as the stagnation temperature in terms of the mach number (formula in notes)
  - At very high velocities, the stagnation temperature can be significantly higher than the free stream temperature
  - Shockwaves will dissipate some of this temperature
- The rule of thumb is we must take compressibility into account when the density changes are greater than 5%
  - If we use the formula of $\frac{\rho_0}{\rho}$ from M, we get a mach number of about 0.3 as the critical threshold

Variation of Fluid Velocity With Flow Area

- Using the continuity equation and conservation of energy we can derive $\frac{\mathrm{d}A}{A} = -\frac{\mathrm{d}V}{V}(1 - M^2)$
  - In subsonic flow, we get $\frac{\mathrm{d}A}{\mathrm{d}V} < 0$ – velocity decreases with increasing area
    * We know this from the incompressible Bernoulli equation
  - In supersonic flow, $\frac{\mathrm{d}A}{\mathrm{d}V} > 0$ – velocity increases with increasing area
    * Density rapidly decreases so the air fills the entire channel
  - In sonic flow, $\frac{\mathrm{d}A}{\mathrm{d}V} = 0$ which means $\mathrm{d}A = 0$
    * If we have area as a function of x, then we have $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$, which means the point at which we have sonic flow must be either a point of maximum or minimum area
    * Since $\frac{\mathrm{d}A}{\mathrm{d}V} < 0$ for subsonic flow, the only way we can get this is to have a converging-diverging duct; then we get $M = 1$ at the minimum point of the duct (the throat)
    * After the throat, the flow can accelerate to supersonic (this depends on the duct design and the pressure after the duct)