Lecture 31, Dec 6, 2021

Simple Harmonic Oscillator Energy

- Recall $a(t) = -A\omega^2 \cos(\omega t + \phi) \implies F_x = -mA\omega^2 \cos(\omega t + \phi) = -m\omega^2 x(t)$
- We can integrate this force to find the work done: $W = \int_{x_0}^{x} -m\omega^2 x \,\mathrm{d}x = \frac{1}{2}m\omega^2(x_0^2 - x^2)$
- $W = \Delta K \implies \Delta K = \frac{1}{2}m\omega^2(x_0^2 - x^2)$
- If we make the oscillator a closed system then $\Delta U = -\Delta K = \frac{1}{2}m\omega^2(x^2 - x_0^2)$
  - Since potential at x_0 is arbitrary we can set this to 0 so $\Delta U = \frac{1}{2}m\omega^2 x^2$
- The total energy is then $E = U + K = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}m\omega^2 A^2$
  - The energy of a simple harmonic oscillator is constant and it only trades potential and kinetic energy back and forth
  - The energy is proportional to the square of the amplitude

Torsional Oscillators

- A disk suspended by a twisting fibre creates a torsional oscillator in simple harmonic motion
  - $\tau = I\alpha$
  - For small angular displacements $\tau = -\kappa(\omega - \omega_0)$ where κ is the angular equivalent of k
  - The differential equation is then $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{\kappa}{I}\theta$ which is the same as the one for translational simple harmonic motion, just with the translational terms substituted by rotational ones
- The equations of motion are identical to translational simple harmonic motion $\theta = \theta_{max}\cos(\omega t + \phi)$ and $\omega = \sqrt{\frac{\kappa}{I}}$

Examples of Oscillating Systems

- All simple harmonic oscillators obey $\frac{\mathrm{d}^x t}{\mathrm{d}+^x}\omega^2 x = 0$
- Examples:
  1. Mass on a spring: $m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0 \implies \omega^2 = \frac{k}{m}$
  2. Torsional oscillator: $I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \kappa\theta = 0 \implies \omega^2 = \frac{\kappa}{I}$
  3. Pendulum: $ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl\sin\theta \approx ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl\theta = 0 \implies \omega^2 = \frac{g}{l}$
  4. Floating object bobbing in water: $m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + g(\rho A y) = 0 \implies \omega^2 = \frac{g\rho A}{m}$
     - A buoyant object will float in the water at some neutral point, and if pushed past this neutral point then $F_y \propto y$
  5. Capacitor-inductor circuit $L\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + \frac{Q}{C} = 0 \implies \omega^2 = \frac{1}{LC}$
     - Voltage drop across capacitor: $\frac{Q}{C}$
     - Voltage drop across inductor: $L\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2}$

Damped Oscillations

- In reality there is always some friction present, causing the oscillator to lose energy and thus amplitude

- The loss in amplitude is called *damping*
- Often the damping is caused by viscous friction, with the damping force opposing and proportional to velocity $F = -b\vec{v}$
- Therefore $F_x = -(kx + bv_x) = ma_x \implies m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$
- The solution is given by $x(t) = Ae^{-\frac{bt}{2m}}\cos(\omega_d t + \phi)$ where $\omega_d = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
  - A damped harmonic oscillator oscillates slower than the equivalent undamped oscillator
  - Let the damping time constant $\gamma = \frac{b}{m}$, with units of time
  - The amplitude can be expressed as $x_{max} = Ae^{\frac{-\gamma t}{2}}$
  - The energy remaining is then $E(t) = \frac{1}{2}m\omega^2 x_{max}^2 = \left(\frac{1}{2}\omega^2 A^2\right)e^{-\gamma t}$
- As b increases the oscillator decays faster, and when $b > 2m\omega$ the system is overdamped and there are no oscillations at all

Q Factor

- Q is the quality factor of an oscillator and measures the rate of decay
  - Differs from ω in the sense that Q measures the number of oscillations
- $Q \equiv \frac{\omega}{\gamma} = \frac{2\pi}{\gamma T}$
- If $Q = 2\pi$ then the energy falls to $e^{-1} = 37\%$ of its original energy in a single cycle
- A good bell has $Q = 100$, electronic circuits have $Q = 10^6$, quantum systems an have $Q = 10^9$