Lecture 30, Dec 2, 2021

Approximation of Linear Restoring Forces

- Simple harmonic motion occurs whenever there is a linear restoring force
- Usually restoring forces can be approximated linearly even when they're not (for small displacements)
- This means that there is usually simple harmonic motion occurring whenever there is a point of stable equilibrium (forces attempt to restore the object to equilibrium instead of pushing it away or doing nothing when the object goes away from equilibrium)
- If $F_x = -kx$ then $U = -\int F_x \, \mathrm{d}x = \frac{kx^2}{2} + U(0)$
  - $k = \frac{\mathrm{d}^2 U}{\mathrm{d}x^2}$ is the curvature of the potential, and we can derive this approximate k even when the restoring force is not truly linear
  - This can be done for any potential but the motion is only truly harmonic for k very close to equilibrium points
- A pendulum is such an example; the restoring torque is not truly linear but for small angles $\sin x \approx x$
  - Potential is parabolic around the minimum for true harmonic motion
  - Since the pendulum is constrained to circular motion, its potential (w.r.t. angle) is sinusoidal
  - In a simple pendulum there is negligible mass in the string, the pendulum has a force mg pointing straight down from gravity, acting on the mass
  - The radial component is $mg\cos\theta$ and the tangential component is $mg\sin\theta$
    * The radial component is opposed by the tension in the string but the tangential component is not
    * Sum of the forces in the radial direction is $mg\cos\theta - T = ma_r$
    * $a_r = -a_c = r\omega^2$ pointing in the $-\hat{r}$ direction towards the centre
    * Since the pendulum doesn't move radially this means that $mg\cos\theta = T$
    * Forces in the tangent direction is $mg\sin\theta$ without a counterbalancing force, so the tangential acceleration is $r\alpha = la_\theta$
    * $-mg\sin\theta = ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \implies \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mg}{ml}\sin\theta = 0$ is the equation of motion for the simple pendulum
      - This would be the same equation as simple harmonic motion if the $\sin\theta$ was instead θ
    * Behaviour is non harmonic if $\sin\theta$ is not close to θ, when $\frac{\theta^3}{3!}$ is not negligible
      - Expanding this out the contribution is $\frac{\frac{\theta^3}{3!}}{\theta}$, so we want $\frac{\theta^2}{6} \ll 1$
  - In a physical pendulum in which the mass of the rod cannot be ignored, we can use I instead of m and now the gravitational force acts on the centre of mass of the entire rod-mass system
    * The torque applied is $mgl\sin\theta$ where l is the distance of the centre of mass of the system from the pivot
    * For the physical pendulum $\sum \tau = -mgl\sin\theta = I\alpha_\theta = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mgl\sin\theta}{I} = 0$
      - If $I = ml^2$ then this would be the same as the simple pendulum equation
  - Compare to simple harmonic motion $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0$
  - Even for a spring, $F = -kx$ is only an approximation
- For any arbitrary (differentiable) potential we can always approximate its behaviour at minima using parabolas, which lead to simple harmonic motion