Lecture 24, Nov 15, 2021

Work in Multiple Dimensions

- In multiple dimensions power is generalized from $P = Fv$ to $P = \vec{F} \cdot \vec{v}$
- Power is the rate of change of mechanical energy
- In one dimension $K = \frac{1}{2}mv^2 \implies \frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}v^2 = mv\frac{\mathrm{d}v}{\mathrm{d}t} = mva = Fv$
- In multiple dimensions $K = \frac{1}{2}m\|\vec{v}\|^2$ where $\|\vec{v}\|^2 = v_x^2 + v_y^2 + v_z^2$, so $\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{v}\|^2 = \frac{1}{2}m(2v_xa_x + 2v_ya_y + 2v_za_z) = F_xv_x + F_yv_y + F_zv_z = \vec{F} \cdot \vec{v}$
- Using the dot product $\vec{F} \cdot \vec{v} = F_xv_x + F_yv_y + F_zv_z = \left\|\vec{F}\right\|\|\vec{v}\|\cos\theta$, which computes the velocity of the point of application in the direction of the applied force
  - Example: The earth orbiting around the sun does no work, because the gravitational force pulling it inwards is perpendicular to the displacement of the Earth
- Generalizing this to work, $\mathrm{d}W = P\,\mathrm{d}t = \vec{F} \cdot \Delta\mathrm{d}\vec{r}$ and taking the integral we obtain work as a line integral: $W = \int \vec{F} \cdot \mathrm{d}\vec{r}$
  - To actually compute this integral we need to parameterize $\vec{r}(t)$, and turn $\mathrm{d}\vec{r}$ into $\begin{bmatrix}\mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z\end{bmatrix}$, each of which expressed in terms of $\mathrm{d}t$, so the integral can now be computed
  - $\int \vec{F} \cdot \mathrm{d}\vec{r} = \int \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t$ and $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ is just the velocity

Force Created by Potential Energy

- Recall that systems always accelerate towards lower potential every
- Since $\mathrm{d}K = W\,\mathrm{d}t = F\,\mathrm{d}x$, thus $\mathrm{d}U = -F\,\mathrm{d}x$ and $F = -\frac{\mathrm{d}U}{\mathrm{d}x}$
- So if we know the change in potential energy then we know the force
- In multiple dimensions $F_x = -\frac{\partial U}{\partial x}, F_y = -\frac{\partial U}{\partial y}$, etc
- If we write force as a vector then $\vec{F} = -\frac{\partial U}{\partial x}\hat{i} + -\frac{\partial U}{\partial y}\hat{j}$, which is the *gradient* of U, $\vec{F} = -\vec{\nabla}U$, thus force is the gradient of potential
  - e.g. gravitational potential $U(x, y) = mgy \implies \vec{F} = \begin{bmatrix}0 \\ -mg\end{bmatrix}$, or a spring $U(x, y) = \frac{1}{2}kx^2 \implies \vec{F} = \begin{bmatrix}-kx \\ 0\end{bmatrix}$
- Forces derived from the gradient of a potential are *conservative*; $U + K$ is always conserved
- With a conservative force the work done is independent of the path taken, by the fundamental theorem of calculus for line integrals