Lecture 12, Oct 6, 2021

Preferred Reference Frames (Centre of Momentum)

- Recall that from the previous lecture that for two observers in inertial reference frames, all physics remains the same
- Try to find an inertial reference frame where total momentum is 0
- Consider 3 objects of mass 1, 2, 3kg moving at 1, 2, 3m/s from frame A
  - $p_A = m_1 v_1 + m_2 v_2 + m_3 v_3$
  - $p_F = m_1(v_1 - v_{AF}) + m_2(v_2 - v_{AF}) + m_3(v_3 - v_{AF})$ which is a linear function of v_{AF}
    - At some value of $v_{AF} = v_{AZ}$, $p_F = 0$; how do we calculate this value?
  - $p_A = \sum m_i v_i \to p_Z = \sum m_i(v_i - v_z) = \sum m_i v_i - \sum m_i v_z = \sum m_i v_i - v_z M_{tot}$
  - $p_{tot} = 0 \implies v_z = \dfrac{\sum m_i v_i}{M_{tot}} = \dfrac{p_A}{M_{tot}}$
  - Note here v_z is relative to A
  - Akin to a weighted average for centre of gravity

Centre of Mass

- If we integrate our expression for v_z, we get the mass-weighted average position, i.e. the center of mass
- The centre-of-mass velocity $\vec{v}_{cm}$ is exactly the velocity of the zero-momentum reference frame
- When measured from a reference frame moving the same velocity as the centre of mass of the system, the system's total momentum is 0
- During a collision, the momentum of the system and thus the zero-momentum reference frame velocity does not change, and so the velocity of the centre of mass of the system does not change

Kinetic Energy in Different Inertial Frames

- $K_A = \sum \frac{1}{2} m_i v_i^2$
- $K_F = \sum \frac{1}{2} m_i (v_i - v_F)^2 = \frac{1}{2} \sum m_i v_i^2 - \sum m_i v_i v_F + \frac{1}{2} m_i v_F^2 = K_A - M_{tot} v_z v_F + \frac{1}{2} M_{tot} v_F^2$
- $\frac{M}{2}(v_F - v_z)^2 = \frac{M}{2} v_F^2 - M v_F v_z + \frac{M}{2} M v_F^2$
- K_{min} is the kinetic energy as observed at v_z, the centre of momentum frame; all other observers see more kinetic energy
- In summary, in the frame moving at $v_{AF} = v_{cm}$, the centre of mass is stationary, the total momentum remains zero, and the kinetic energy is minimized
- What is the physical significance of K_{cm}?
  - All kinetic energy from K_i is converted to some ΔE_{int}
  - $\Delta E_{int} = K_i$ for $e = 0$ in the z-frame
  - In a totally inelastic collision the two objects stick together, and if observed at the center of momentum this means they both stop
  - In the center of momentum frame K_{min} goes all the way to zero, but in all other frames there is some kinetic energy left over

Convertible Kinetic Energy (Textbook)

- For a system of objects, the kinetic energy as measured in the Earth reference frame is K_E
- $K_E = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i (v_{cm} + v_{Zi}) = \frac{1}{2} v_{cm}^2 \sum m_i + v_{cm} \sum m_i v_{Zi} + \frac{1}{2} \sum m_i v_{Zi}^2$
  - From the definition of the zero-momentum reference frame we know that $\sum m_i v_{Zi} = \sum p_{Zi} = 0$ so the middle term disappears
  - $\frac{1}{2} \sum m_i v_{Zi}^2$ is just the sum of all kinetic energies as observed in the zero-momentum reference frame K_Z

  - If we substitute $m = \sum m_i$ we get $K_E = \frac{1}{2}mv_{cm}^2 + K_Z$
  - The $\frac{1}{2}mv_{cm}^2$ term is the kinetic energy associated with the motion of the center of mass of the system K_{cm}
    * For an isolated system, K_{cm} is *nonconvertible*, i.e. cannot be converted into internal energy, because it is needed to preserve conservation of momentum; recall how the velocity of the centre of mass of a system does not change after a collision
  - The K_Z term is the system's *convertible kinetic energy* K_{conv}, which can be converted into internal energy while conserving the momentum of the system
- Therefore we have $K = K_{conv} + K_{cm}$, where $K_{conv} = \frac{1}{2}\sum m_i v_i - \frac{1}{2}mv_{cm}^2$ and this holds in any inertial reference frame; i.e. the kinetic energy of any system can be split into a convertible and a nonconvertible part
- We can also express $K_{conv} = \frac{1}{2}\mu v_{12}^2$, where μ is the *reduced inertia/mass*
- In a totally inelastic collision, all the convertible kinetic energy is converted into internal energy, but the system has to retain its nonconvertible kinetic energy K_{cm} to maintain momentum conservation