Lecture 1, Sep 9, 2021

Classical Mechanics: Overview

- The study of objects in motion and the influence of forces on them
- Course involves review using calc and more advanced topics, symmetries of situations
  - Learn the difference between hypotheses, assumptions, and derived relations
- Physics is about testing falsifiable hypotheses: learn by disproving hypotheses; uncertainties are very important

Course Resources

- Quercus homepage (course materials incl. visuals, problem sets, grades)
  - Primary source: Link
- Piazza (online discussion forum between peers and TAs)
- Textbook and MyMastering (eText: Principles and Practices of Physics (Mazur))
  - Includes problem sets for grades and practice problems not for grades
  - Setup: Link

Grade Breakdown

- 10% Problem sets
- 25% Labs
- 30% Term tests (early Oct, mid Nov)
- 35% Final exam

TODO:

- [x] Set up Piazza
- [x] Set up MyMastering
- [x] Read ch 1-3 in textbook (focus on ch 2-3)
- [x] Problem set PS0 (due Sep 16, 11:59pm)

First Deliverables

- PS0 due Sep 16
- PS1 due Sep 24
- First lab writeup due early Oct

Lecture 2, Sep 13, 2021

- Scientific method: Observations induce hypothesis, hypothesis deduce prediction, prediction tests observations
  - Induction makes generalizations, deductions applies the hypothesis to specific cases
  - Hypothesis is a combination of a model and assumptions
- Physical laws arise from symmetries:
  1. Time invariance: Is physics different "now" compared to "then"?
     - Energy conservation $\iff$ time invariance
  2. Spatial invariance: Is physics different "here" compared to "there"?
     - Momentum conservation $\iff$ spatial invariance
  3. Rotational invariance: Is physics different from "this perspective" compared to a rotated perspective?
     - Angular momentum conservation $\iff$ rotational invariance
- Noether's theorem: Each symmetry corresponds to a conservation law

- Quantities such as m are typically in italics, and units such as m are typically in roman

Lecture 3, Sep 15, 2021

Motion in 1D

The Calculus of Motion

- The calculus hierarchy: $x(t) \underset{\int \mathrm{d}t}{\overset{\frac{\mathrm{d}}{\mathrm{d}t}}{\rightleftharpoons}} v(t) \underset{\int \mathrm{d}t}{\overset{\frac{\mathrm{d}}{\mathrm{d}t}}{\rightleftharpoons}} a(t)$
  - Integration only tells you how much the functions changed, not where they started
- Essential calculus: $\frac{\mathrm{d}}{\mathrm{d}t}t^n = nt^{n-1} \implies \int nt^{n-1}\,\mathrm{d}t = t^n + C \implies \int t^n\,\mathrm{d}t = \frac{1}{n+1}t^{n+1}$
- Example: $x(t) = 2[m] + 3\left[\mathrm{m/s^3}\right]t^3$ (square brackets denote units)
  - $v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(2 + 3t^3\right) = 0 + 9t^2$
- Example: $v(t) = 3\,[\mathrm{m/s}]$ find $x(t)$ for $x(0) = 2\,[\mathrm{m}]$
  - $x(0)$ is the constant of integration
- When differentiating by time, a unit of time is introduced in the denominator; e.g. $\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{m}] = [\mathrm{m/s}]$

Units of Motion

Quantity	Symbol	Units
time	t	T
position	x or $\vec{r}$ in multiple dimensions	L
velocity	v or $\vec{v}$ in multiple dimensions	L/T
acceleration	a or $\vec{a}$ in multiple dimensions	$\mathrm{L/T^2}$
speed	$\lvert v\rvert$ or $\lVert\vec{v}\rVert$ in multiple dimensions	L/T

Lecture 4, Sep 16, 2021

One Dimensional Motion, Continued

- $a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t}$ is only defined for smooth and continuous functions; when collecting real world data there could be sharp changes in velocity that may not be differentiable (infinite change/discontinuity in acceleration)
  - In reality the actual motion between points is still continuous but the sampling is too rough to tell
  - Can't be described by a simple line

Definite Integrals

- $\int_a^b f(t)\,\mathrm{d}t$, the definite integral or area under the curve, can be calculated from the indefinite integral $\int f(t)\,\mathrm{d}t$, from the fundamental theorem of calculus
  - Definite integral is a value; indefinite integral is a function
  - $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$ where $F(t) = \int f(t)\,\mathrm{d}t$

Velocity and Acceleration

- $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $x = \int v\,\mathrm{d}t$

- $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ and $v = \int a \, \mathrm{d}t$
- Therefore if we have the velocity and need to know the change in position between two points in time, use $\int_{t_1}^{t_2} v(t) \, \mathrm{d}t = F(t_2) - F(t_1) = x_2 - x_1 = \Delta x$; similarly for velocity/acceleration

Constant Velocity Equations

- $v(t) = v_0$
- $\Delta x = \int_{t_0}^{t_f} v(t) \, \mathrm{d}t = v_0 \Delta t$

Constant Acceleration Equations

- $a(t) = a_0$
- $\Delta v = \int_{t_0}^{t_f} a(t) \, \mathrm{d}t = a_0 \Delta t$
- $v(x) = \Delta v + v_0 = a_0 \Delta t + v_0$

Lecture 5, Sep 20, 2021

Projectile Motion in 1D

- "Projectile": any object launched with some initial velocity v_0; modelled by $x_f = x_i + v_i t_f + \frac{1}{2} a t_f^2$
- A projectile launched upwards with v_i and a projectile launched downwards with $-v_i$ have the same downward speed when the projectile passes through the starting point
  - $x_f = x_i \implies v_i t + \frac{1}{2} a t^2 = 0$ and $v(t) = v_i - gt \implies t = \frac{v(t) - v_i}{-g} = \frac{v_i - v}{g} \implies 0 = v_i \left(\frac{v_i - v}{g} \right) - \frac{1}{2} g \left(\frac{v_i - v}{g} \right)^2 = \left(\frac{v_i - v}{g} \right) \left[v_i - \frac{1}{2} g \left(\frac{v_i - v}{g} \right) \right] = \left(\frac{v_i - v}{g} \right) \left[v_i - \frac{1}{2} (v_i - v) \right]$, zero when $v = -v_i$ or $v = v_i$

Inclined Planes and Free Fall

- Galileo observed that the ratio $\frac{x_i}{t_i^2}$ was constant; i.e. the position is proportional to time squared, when the object is rolling down an inclined plane
  - This ratio is a function of theta: $ax = g \sin \theta$
- When the plane is at 90°, the object is in free fall

Instantaneous Acceleration

- $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right) \equiv \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$
- The instantaneous acceleration is the "curvature" of the position function (related? equal to? the actual curvature $\kappa = \frac{1}{R}$)

Lecture 6, Sep 22, 2021

Collisions and Inertia

- Consider two identical carts on an airtrack undergoing elastic collision; in an ideal work the two carts will always exchange speeds, regardless of their initial speeds

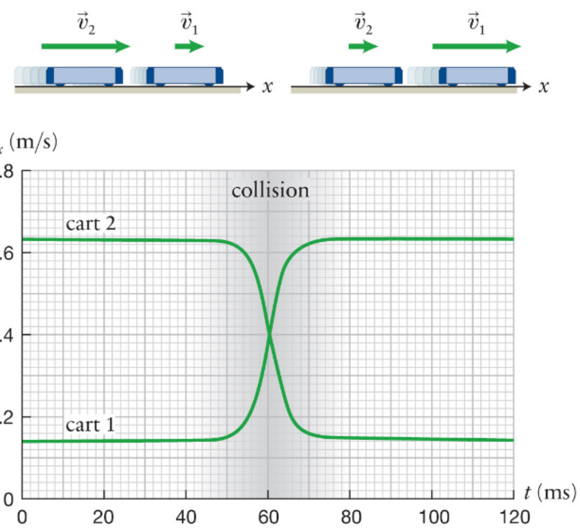

Figure 1: colliding carts

- If one cart is twice as heavy as the other, after the collision the lighter cart will lose twice as much speed as the heavier cart (possibly reversing)
- From this we can observe that inertia is proportional to the mass of the cart; we can also determine the inertia of an object by colliding it with a known mass
  - The ratio of the velocity changes is the inverse ratio of the masses; e.g. if object 2 had a velocity change that is twice as much as object 1, then it is half as massive
- If friction is introduced, the straight parts have a downward slope determined by the friction

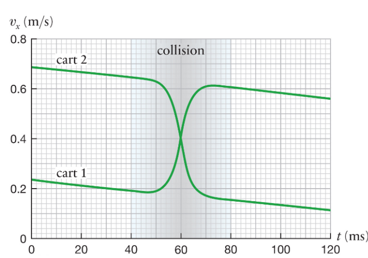

Figure 2: colliding carts with friction

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  - This might differ depending on the type of friction, e.g. kinetic vs viscous friction
- With this we can define inertia: $\dfrac{m_u}{m_s} \equiv -\dfrac{\Delta v_{sx}}{\Delta v_{ux}} \implies m_u \equiv -\dfrac{\Delta v_{sx}}{\Delta v_{ux}} m_s$
  - A one kilogram mass is the inertial standard m_s
  - **Mass is a way to measure inertia/mass is inertia**

Momentum

- We can rearrange the equations to get $m_u \Delta v_u + m_s \Delta v_s = 0$, leading us to define *momentum* as $p_x \equiv m v_x$
- Since $m_u \Delta v_u + m_s \Delta v_s = 0 \implies m_u(v_{uf} - v_{ui}) + m_s(v_{sf} - v_{si}) = 0 \implies m_u v_{ui} + m_s v_{si} = m_u v_{uf} + m_s v_{sf}$, momentum is conserved; change in momentum of s is balanced by change in momentum of u
  - $\Delta p_u + \Delta p_s = 0 \iff p_{ui} + p_{si} = p_{uf} + p_{sf}$

- We can use this formula and go back to the carts before (s is the standard cart, d is the double cart): $\Delta p_s + \Delta p_d = 0 \implies m_s \Delta v_s + m_d \Delta v_d \implies \Delta v_d = -\frac{m_s}{m_d} \Delta v_s = -2\Delta v_s$

Lecture 7, Sep 23, 2021

Momentum Continued

- Momentum is a vector: $\vec{p} = m\vec{v}$ with units of $\text{kg} \cdot \text{m/s}$
- The momentum of a system is the sum of all the momentum of its pieces, and this quantity is conserved
- Momentum is only conserved inside an isolated system, i.e. all the pieces in the interaction are included
  - e.g. momentum is conserved in a collision if both carts are in the system, but it is not if only one cart is in the system, since the other cart would be an external
- The change in momentum is impulse: $\Delta \vec{p} = \vec{J}$ (this quantity is only nonzero when external forces act on the system)
- This even works when the speed of particles is near c, such as in Compton scattering
- To choose the right system, identify the interactions, then eliminate objects if interactions cause no acceleration
  - e.g. for two carts sliding without friction, the surface is eliminated, since the interaction with the surface does not create a force/acceleration
- When the two objects stick together, the momentum is still conserved
- If the two carts stick together, they will both move together at the same slower velocity
- $\Delta p_1 + \Delta p_2 = 0 \implies m_1 \Delta v_1 + m_2 \Delta v_2 = 0 \implies \frac{\Delta v_2}{\Delta v_1} = -\frac{m_1}{m_2} \implies \Delta v_2 = -\frac{m_1 \Delta v_1}{m_2}$

Lecture 8, Sep 27, 2021

Energy

- Energy comes in many forms: electromagnetic waves, rotational/translational kinetic, thermal, bio-chemical, etc
- We will study mechanical energy: translational and rotational kinetic energy

Classifying Collisions

- Elastic collisions: Δv_{12} initial $= \Delta v_{12}$ final $\equiv |v_2 - v_1|$
  - The difference in velocity remains the same before and after the collision, even if the two masses are different
- Totally inelastic collisions: Δv_{12} final $= 0$, i.e. the two objects stick together

Kinetic Energy

- $K = \frac{1}{2}mv^2$ and does not depend on the direction of motion
- Kinetic energy is conserved in elastic collisions but **not** in inelastic collisions, whereas momentum is conserved in both
- Inelastic collisions usually result in some sort of irreversible change, e.g. irreversible deformations
  - The energy lost goes into the internal energy E_{int}: $E = K + E_{int}$
- A closed system has its energy conserved, an open system does not
  - Closed/open for energy, isolated/not isolated for momentum
  - Remember momentum is a vector, so even if the speed does not change, if the direction of motion changes, momentum changes; therefore orbiting planets are not isolated (they change direction and thus momentum)

Lecture 9, Sep 29, 2021

Identifying and Choosing Closed Systems

- A closed system is any system that does not transfer energy in/out of it
- Identify all objects that change state or state of motion, and group them together to make a closed system

Elastic Collisions in Isolated and Closed Systems

- The difference in speed remains the same after an elastic collision
- Relative speed remains the same but relative velocity is negated

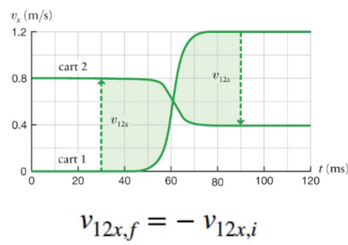

Figure 3: relative speed

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- Conservation of momentum, conservation of energy and elastic collisions can be derived from each other
  - $$\begin{aligned} & k_i = k_f \\ \implies & \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \\ \implies & m_1(v_{1f}^2 - v_{1i}^2) = m_2(v_{2f}^2 - v_{2i}^2) \\ \implies & m_1(v_{1f} + v_{1i})(v_{1f} - v_{1i}) = m_2(v_{2f} + v_{2i})(v_{2f} - v_{2f}) \\ \implies & \Delta p_1(v_{1f} + v_{1i}) = \Delta p_2(v_{2f} + v_{2i}) \end{aligned}$$
- In an elastic collision $k_i = k_f$
- Energy is measured in Joules: $1\text{J} = 1\text{kg}\cdot\text{m}^2/\text{s}^2 = 1\text{m}\cdot\text{kg}\cdot\text{m}/\text{s}^2 = 1\text{N}\cdot\text{m}$

Quantifying (In)Elastic Collisions

- We can quantify how elastic a collision is with the *coefficient of restitution*: $e \equiv \dfrac{v_{12f}}{v_{12i}}$
- The coefficient of restitution is the ratio between the final difference in *speed* and initial difference in *speed*
- The coefficient of restitution is positive, even though the direction of velocity reverses in an elastic collision
- Elastic collisions have $e = 1$, while totally inelastic collisions have $e = 0$

Lecture 10, Sep 30, 2021

Coefficient of Restitution

- $e = 1 + \dfrac{2\Delta E_{int}}{m_1 \Delta v_1 v_{2,1i}}$ (from $K_i = \Delta E_{int} + K_f$)
  - $v_{2,1i}$ is always less than zero

* If the two carts are heading towards each other then $v_{2,1i} = v_{1i} - v_{2i}$ is positive, but Δv_1 is negative since cart 1 slows down
  - Since the term on the right is always less than zero, any $\Delta E_{int} \neq 0$ will result in a e less than 1
  - $e \neq 1$ is a result of change in internal energy $\Delta E_{int} \neq 0$
  - When $\Delta E_{int} < 0$, $e > 1$, and this is an explosive separation (energy is introduced into the system)

Explosive Separations

- Cases where $e > 1$, kinetic energy is introduced (from the internal energy of the explosive/spring/etc)

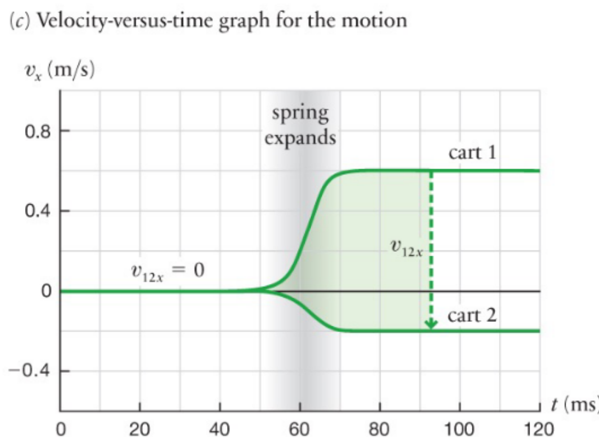

Figure 4: explosive separation

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- Solve for v_{1f} and v_{2f} with $v_{1i} = v_{2i} = 0$
  - $E_{spring} = -\Delta E_{int}$
  - $$\begin{aligned} & \Delta K + \Delta E_{int} = 0 \\ \implies & K_f - K_i + \Delta E_{int} = 0 \\ \implies & K_f = -\Delta E_{int} \\ \implies & K_f = E_{spring} \\ \implies & \frac{1}{2}(m_1 v_{1f}^2 + m_2 v_{2f}^2) = E_{spring} \end{aligned}$$
  - $$\begin{aligned} & m_1 v_{1f} + m_2 v_2 f = 0 \\ \implies & v_{2f} = \frac{-m_1}{m_2} v_{1f} \\ \implies & \frac{1}{2}\left(m_1 v_{1f}^2 + m_2 \left(\frac{-m_1}{m_2} v_{1f}\right)^2\right) = E_{spring} \\ \implies & v_{1f}^2 = \frac{2E_{spring}}{m_1 + \frac{m_1^2}{m_2}} \end{aligned}$$

Reference Frames and Relativity

- Note: The relativity we're talking about assumes $v \ll c$
- How does motion vary from different perspectives? Are energy, momentum, etc conserved?
- Right now limited to observers with constant velocity difference
- $\vec{v}_{A,B}$ is the velocity of A as observed by B

- Define all speeds relative to an observer: $\vec{v}_{o,r} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}_o - \vec{r}_r) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{r}_{r\to o}$ (reference r, object o)
- Physics that work regardless of perspective:
  - Velocity of objects ✗
  - Momentum of objects ✗(because velocities are different)
  - Total kinetic energy ✗(because velocities are different)
  - Where things happen ✗
  - When things happen ✓(but not in special relativity)
  - Relative velocities ✓
  - Change in momentum ✓
  - Loss of kinetic energy ✓(because they agree on Δv)
  - Increase of internal energy ✓
  - Conservation of momentum ✓
  - Conservation of energy ✓
- Change in kinetic energy holds across perspectives
  - $k_1 = \frac{1}{2}m_1 v_1^2, k_2 = \frac{1}{2}m_2 v_2^2$
  - Observers will agree on Δv_1 and Δv_2
  - Frame A:
    - $\Delta K_A = K_{A,f} - K_{A,i} = \frac{1}{2}m_1(v_{A1f}^2 - v_{A1i}^2)$

Lecture 11, Oct 4, 2021

Inertial Reference Frames

- An *inertial reference frame* is a reference frame moving at a constant speed; two observers moving relative to one another at a constant speed are both in inertial reference frames
  - However, both observers could be accelerating relative to some third frame (e.g. the Earth); as long as they're accelerating the same amount and so have a constant relative speed
    - e.g. two astronauts orbiting the earth
- Example of noninertial reference frame: accelerating car
  - Noninertial reference frames lead to fictitious forces from the perspective of the observer in the reference frame, e.g. centrifugal force

Galilean Coordinate Transformations

- The core of Galilean relativity is that all observers agree on the time of events: $t_{Be} = t_{Ae} = t_e$
- The position of events are different: $\vec{r}_{Be} = \vec{r}_{Ae} - \vec{v}_{AB}t_e$
- Note: First letter is observer, e.g. $\vec{r}_{AB}$ is the position of B relative to A
- Suppose A and B are in inertial reference frames and c is an accelerating object
  - $\Delta\vec{r}_{Ac} = \Delta\vec{r}_{AB} + \Delta\vec{r}_{Bc}$
  - $\Delta\vec{r}_{Ac} = \Delta\vec{r}_{AB} + \Delta\vec{r}_{Bc} \implies \vec{v}_{Ac} = \vec{v}_{AB} + \vec{v}_{Bc} \implies \Delta\vec{v}_{Ac} = \Delta\vec{v}_{Bc}$ assuming $\vec{v}_{AB}$ is constant
  - Therefore $\vec{a}_{Ac} = \vec{a}_{Bc}$, assuming $\vec{v}_{AB}$ is constant
- If we're careful about the subscripts then we can use "cancellation": $\vec{r}_{Ae} = \vec{r}_{A\cancel{B}} + \vec{r}_{\cancel{B}e}$
- Position vectors are each other's opposites: $\vec{r}_{AB} = -\vec{r}_{BA}$, which also applies to velocities, accelerations, etc

Principles of Relativity

- Since changes in velocity are the same regardless of inertial reference frame, momentum and kinetic energy are conserved
  - Note only *changes* in kinetic energy are the same, but the entire kinetic energy may be different
  - Differences in kinetic energy are the same even for inelastic collisions
- In general, *all laws of physics are frame independent*
- As a direct result, physics measurements cannot distinguish one inertial reference frame from another

Lecture 12, Oct 6, 2021

Preferred Reference Frames (Centre of Momentum)

- Recall that from the previous lecture that for two observers in inertial reference frames, all physics remains the same
- Try to find an inertial reference frame where total momentum is 0
- Consider 3 objects of mass 1, 2, 3kg moving at 1, 2, 3m/s from frame A
  - $p_A = m_1v_1 + m_2v_2 + m_3v_3$
  - $p_F = m_1(v_1 - v_{AF}) + m_2(v_2 - v_{AF}) + m_3(v_3 - v_{AF})$ which is a linear function of v_{AF}
    - At some value of $v_{AF} = v_{AZ}$, $p_F = 0$; how do we calculate this value?
  - $p_A = \sum m_iv_i \to p_Z = \sum m_i(v_i - v_z) = \sum m_iv_i - \sum m_iv_z = \sum m_iv_i - v_zM_{tot}$
  - $p_{tot} = 0 \implies v_z = \dfrac{\sum m_iv_i}{M_{tot}} = \dfrac{p_A}{M_{tot}}$
  - Note here v_z is relative to A
  - Akin to a weighted average for centre of gravity

Centre of Mass

- If we integrate our expression for v_z, we get the mass-weighted average position, i.e. the center of mass
- The centre-of-mass velocity $\vec{v}_{cm}$ is exactly the velocity of the zero-momentum reference frame
- When measured from a reference frame moving the same velocity as the centre of mass of the system, the system's total momentum is 0
- During a collision, the momentum of the system and thus the zero-momentum reference frame velocity does not change, and so the velocity of the centre of mass of the system does not change

Kinetic Energy in Different Inertial Frames

- $K_A = \sum \frac{1}{2}m_iv_i^2$
- $K_F = \sum \frac{1}{2}m_i(v_i - v_F)^2 = \frac{1}{2}\sum m_iv_i^2 - \sum m_iv_iv_F + \frac{1}{2}m_iv_F^2 = K_A - M_{tot}v_zv_F + \frac{1}{2}M_{tot}v_F^2$
- $\frac{M}{2}(v_F - v_z)^2 = \frac{M}{2}v_F^2 - Mv_Fv_z + \frac{M}{2}Mv_F^2$
- K_{min} is the kinetic energy as observed at v_z, the centre of momentum frame; all other observers see more kinetic energy
- In summary, in the frame moving at $v_{AF} = v_{cm}$, the centre of mass is stationary, the total momentum remains zero, and the kinetic energy is minimized
- What is the physical significance of K_{cm}?
  - All kinetic energy from K_i is converted to some ΔE_{int}
  - $\Delta E_{int} = K_i$ for $e = 0$ in the z-frame
  - In a totally inelastic collision the two objects stick together, and if observed at the center of momentum this means they both stop
  - In the center of momentum frame K_{min} goes all the way to zero, but in all other frames there is some kinetic energy left over

Convertible Kinetic Energy (Textbook)

- For a system of objects, the kinetic energy as measured in the Earth reference frame is K_E
- $K_E = \frac{1}{2}\sum m_iv_i^2 = \frac{1}{2}\sum m_i(v_{cm} + v_{Zi}) = \frac{1}{2}v_{cm}^2\sum m_i + v_{cm}\sum m_iv_{Zi} + \frac{1}{2}\sum m_iv_{Zi}^2$
  - From the definition of the zero-momentum reference frame we know that $\sum m_iv_{Zi} = \sum p_{Zi} = 0$ so the middle term disappears
  - $\frac{1}{2}\sum m_iv_{Zi}^2$ is just the sum of all kinetic energies as observed in the zero-momentum reference frame K_Z

- If we substitute $m = \sum m_i$ we get $K_E = \frac{1}{2}mv_{cm}^2 + K_Z$
- The $\frac{1}{2}mv_{cm}^2$ term is the kinetic energy associated with the motion of the center of mass of the system K_{cm}
  - For an isolated system, K_{cm} is *nonconvertible*, i.e. cannot be converted into internal energy, because it is needed to preserve conservation of momentum; recall how the velocity of the centre of mass of a system does not change after a collision
- The K_Z term is the system's *convertible kinetic energy* K_{conv}, which can be converted into internal energy while conserving the momentum of the system

- Therefore we have $K = K_{conv} + K_{cm}$, where $K_{conv} = \frac{1}{2}\sum m_i v_i - \frac{1}{2}mv_{cm}^2$ and this holds in any inertial reference frame; i.e. the kinetic energy of any system can be split into a convertible and a nonconvertible part
- We can also express $K_{conv} = \frac{1}{2}\mu v_{12}^2$, where μ is the *reduced inertia/mass*
- In a totally inelastic collision, all the convertible kinetic energy is converted into internal energy, but the system has to retain its nonconvertible kinetic energy K_{cm} to maintain momentum conservation

Lecture 13, Oct 13, 2021

Inelastic Collision From Other Frames

- Kinetic energy of f as observed by A is $K_f = K_{cm} = \frac{1}{2}m(v_{Af} - v_{AZ})^2$; true for both initial and final states in a collision
- $k_{f_i} = K_{cm,i} + \frac{1}{2}m(v_{Af} - v_{cm})^2$
- $k_{f_f} = K_{cm,f} + \frac{1}{2}m(v_{Af} - v_{cm})^2$
- The same amount of kinetic energy can be converted into internal energy in all frames
- The additional kinetic energy as observed from other frames cannot be converted to internal energy even in totally inelastic collisions
  - This is the *unavailable/unconvertible* kinetic energy, and is required for momentum conservation

Convertible Kinetic Energy

- $|\Delta k| = \Delta E_{int}$ since in the center of momentum frame (Z) we can convert all the kinetic energy as there is no momentum
- We can convert this to other frames by adding an additional term for $K_{F,tot} = K_{Z,tot} + \frac{1}{2}m_{tot}(v_{cm,f})^2$
  - The $K_{Z,tot}$ is the convertible part and $\frac{1}{2}m_{tot}(v_{cm,f})^2$ is not convertible
- $K_{conv} = K_{f,tot} - \frac{1}{2}m_{tot}v_{cm,f}^2$ so we can apply a correction to see how much energy is convertible by subtracting off the kinetic energy of the center of momentum frame
- Application: Particle colliders
  - If you collide particles against a fixed target, a large part of the kinetic energy is unconvertible since the momentum needs to be maintained
  - However, if you collide particles against a moving particle such that the collision is in the centre of momentum frame, all the kinetic energy is now convertible

Reduced/Effective Mass

- K_{conv} is a frame independent quantity; how is this related to v_{12}, the relative velocity of the two participants in the collision, which is also frame independent?

- $K_{conv} = K_{tot} - \frac{1}{2}m_{tot}v_{cm}^2$
$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_{tot}\left(\frac{m_1v_1 + m_2v_2}{m_{tot}}\right)^2$$
- $2m_{tot}K_{conv} = (m_1 + m_2)(m_1v_1^2 + m_2v_2^2) - (m_1v_1^2 + m_2v_2^2 + 2m_1m_2v_1v_2) = m_1m_2(v_2 - v_1)^2$
- $K_{conv} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(v_2 - v_1)^2 = \frac{1}{2}\mu v_{12}^2$ where μ is $\frac{m_1m_2}{m_1 + m_2}$ which is the *reduced/effective mass*

Interactions

- In an isolated system, the total momentum before, after, and during a collision stays the same, but the kinetic energy does not
- Since objects don't maintain constant relative speed during the entire collision, the total kinetic energy changes, even for elastic collisions
- If the collision is elastic, the kinetic energy is the same before and after a collision, but it is still decreased during a collision
- At some point during the collision $\Delta v = 0$; there is no way to tell whether the collision is elastic or inelastic by only looking at what happens before this point
- The kinetic energy lost during the collision is converted into internal energy; for elastic collisions some of this energy comes back
- If we're looking at it from the centre of momentum frame, in the middle of the collision all the kinetic energy is converted into internal energy

Lecture 14, Oct 14, 2021

Potential Energy

- During a collision of a cart with a spring, the kinetic energy is converted into potential energy in the spring
- Potential energy is a form of energy that can be converted back into kinetic energy and is associated with reversible changes (e.g. compressing a spring)

Energy Dissipation

- Energy dissipation is the loss of kinetic energy that cannot be reversed
- This usually happens on an atomic scale; the kinetic energy is converted into motion on an atomic scale, and the chaotic and incoherent behaviour of the particles means the energy cannot be converted back into kinetic energy of the cart
- Textbook uses bending vs crumpling a piece of paper

Energy Classification

- The textbook classifies energy into 4 categories:
    1. Kinetic energy: coherent energy associated with motion of objects (easily calculable)
    2. Potential energy: coherent energy associated with configuration of interacting objects (e.g. gravitational, elastic) (easily calculable)
    3. Source energy: incoherent energy used to produce other forms of energy (e.g. chemical, nuclear, solar) (very hard to calculate)
    4. Thermal energy: incoherent energy associated with the chaotic motion of atoms
- Example of the types of energy:
    - Kinetic energy: a litre of gas moving in a car
    - Thermal energy: heating up a litre of gas by 1 degree (usually cannot be converted into kinetic energy)
    - Potential energy: a litre of gas sitting 1m above the floor (can be converted into kinetic energy)

  - Source energy (chemical): Burning a litre of gas (cannot be directly converted into kinetic energy)
- Coherent forms of energy (kinetic, potential) are usually much smaller than the incoherent forms of energy (source, thermal)
- Coherent energy (mechanical energy) consists of kinetic and potential energy; all the atoms are moving in some coherent direction and this can be converted, reversibly, to and from kinetic energy easily
- Incoherent energy consists of thermal and source (e.g. chemical) energy; all the atoms are moving in random directions so this cannot be converted into kinetic energy reversibly and efficiently
- All energy that is not kinetic energy (potential energy, thermal energy, source energy) are all considered internal energy
- Macroscopic objects eventually lose their kinetic energy (e.g. balls bouncing in a box), but on a microscopic level atomic movements do not stop (the subatomic particles are too small to store the kinetic energy in the particles due to the quantum nature of energy)

Interactions and Acceleration

- Momentum conservation requires $\Delta \vec{p_{tot}} = 0 \implies \Delta \vec{p_1} = -\Delta \vec{p_2}$
- Therefore $\frac{\Delta \vec{p_1}}{\Delta t} = -\frac{\Delta \vec{p_2}}{\Delta t} \implies \frac{m_1 \Delta \vec{v_1}}{\Delta t} = -\frac{m_2 \Delta \vec{v_2}}{\Delta t} \implies m_1 \vec{a_1} = m_2 \vec{a_2}$
- Therefore the ratio of accelerations is proportional to the negative inverse of the ratio of masses $\frac{\|\vec{a_1}\|}{\|\vec{a_2}\|} = \frac{-m_2}{m_1}$
- Example: 1000kg car and 2000kg truck both moving at 25m/s towards each other collides in 0.2s
  - $p_i = 1000\text{kg} \cdot 25\text{m/s} - 2000\text{kg} \cdot 25\text{m/s} = -25000\text{kg m/s}$
  - $v_f = \frac{p_f}{m_{tot}} = \frac{p_i}{m_{tot}} = \frac{-25000\text{kg m/s}}{3000\text{kg}} = 8.3\text{m/s}$
  - $a_1 = \frac{-8.3\text{m/s} - 25\text{m/s}}{0.2\text{s}} = -166\text{m/s}^2$ (the car reversed direction so it has a larger acceleration)
  - $a_2 = \frac{-8.3\text{m/s} - (-25\text{m/s})}{0.2\text{s}} = +83\text{m/s}^2$ (the truck continues to move in the same direction so it has a smaller acceleration)
  - The ratio of accelerations $\frac{a_1}{a_2} = -2 = -\frac{2000\text{kg}}{1000\text{kg}} = -\frac{m_2}{m_1}$

Energy Conversion

- A nondissipative (reversible) interaction converts between kinetic and potential energy; all energy is coherent, no energy is dissipated (e.g. zero friction cart hits spring, which does not heat up)
- A dissipative (nonreversible) interaction could convert between kinetic and potential energy and have some energy lost as thermal energy (e.g. car rolling up a hill)
- Another dissipative interaction could convert source energy to all other forms of energy (e.g. burning gas in a car)
- Another dissipative interaction could convert source energy entirely to thermal energy (e.g. burning gas, a rock sitting outside in a sunny day)

Lecture 15, Oct 18, 2021

Interaction Ranges

- All interactions (e.g. magnetic, collisions) have different ranges
- Some interactions are long-range (e.g. magnetic, gravitational) and work without requiring "contact"
- Other interactions are short-range and only work when objects are "contacting" (e.g. collisions)

Fundamental Interactions

- Electromagnetism and gravity can be characterize by fields

- Moving the charges generates waves in the field
- Strong and weak interactions are caused by fundamental particles such as pions
- Out of the 4 fundamental interactions, gravity is the weakest by many orders of magnitude
- Weak interactions cause the spontaneous decay of nuclei

Nondissipative Interactions

- In any closed system $\Delta E = \Delta K + \Delta U + \Delta E_s + \Delta E_{th} = 0$
- For a non-dissipative system $\Delta E_s = \Delta E_{th} = 0$, i.e. energy is only converted between kinetic and potential forms
- The *mechanical energy* of a system $E_{mech} = K + U$ is the sum of the kinetic and potential energy; for a closed, nondissipative system then $\Delta E_{mech} = 0$
- Example: a cart-spring collision is nondissipative; $\Delta K_{cart} = -\Delta U_{spring}$ as the cart slows down and energy is stored in the spring
- Potential energy is only a function of position $U(x)$ and as a result not depend on the path taken (conservative force); we can calculate this function
- Principle of Potential Energy: **The parts of any closed system always tend to accelerate in the direction that lowers the system's potential energy.**

Local Gravitational Potential Energy

- Suppose a ball falls from x_i to x_f
- $\Delta U_G = -\Delta K_{ball}$ and $a_x = -g$
- $\Delta t = \dfrac{-\Delta v_x}{g}$
- $\Delta x = v_i \Delta t - \dfrac{1}{2} g \Delta t^2$, substitute Δt
- $$\begin{aligned}\Delta x &= v_i \frac{-\Delta v_x}{g} - \frac{1}{2} g \left(\frac{\Delta v_x}{g}\right)^2 \\ &= \frac{-\Delta v_x}{2g}\left[2v_{xi} + (v_{xf} - v_{xi})\right] \\ &= \frac{-\Delta v_x}{2g}(v_{xi} + v_{xf}) \\ &= \frac{-(v_{xf} - v_{xi})}{2g}(v_{xi} + v_{xf}) \\ &= -\frac{1}{2g}(v_{xf}^2 - v_{xi}^2)\end{aligned}$$
- $mg\Delta x = -\dfrac{1}{2} m(v_{xf}^2 - v_{xi}^2) \implies mg\Delta x + \Delta K_{ball} = 0$ and since $\Delta U_G + \Delta K_{ball} = 0$, $\Delta U_G = mg\Delta x$

Dissipative Interactions

- With dissipative interactions, the energy is not a function of position (e.g. friction); reactions are nonreversible
- $\Delta E = \Delta K + \Delta U + \Delta E_{th}$, if no reversible potential energy, then $\Delta E = \Delta E_{th} + \Delta K = 0 \implies \Delta K = -\Delta E_{th}$, and E_{th} is not a function of position
- Example: Totally elastic collision where $\Delta K = 0$ has $\Delta E_{th} = 0$; for totally collisions, $\Delta K = \dfrac{1}{2}\mu v_{21,i}(e^2 - 1)$,so $\Delta E_{th} = \dfrac{1}{2}\mu v_{21,i}(1 - e^2)$

Lecture 16, Oct 20, 2021

Momentum and Force

- Force is the rate of change of momentum: $\vec{F} = \frac{\mathrm{d}d}{\mathrm{d}t} m\vec{v}$
- For constant masses this is equal to $\vec{F} = m\vec{a}$
- The acceleration of an object is caused by the net force, i.e. vector sum of all forces on that object
- Since momentum is conserved, $\Delta p_1 = -\Delta p_2 \implies \frac{\Delta p_1}{\Delta t} = \frac{\Delta p_2}{\Delta t} \implies F_1 = -F_2$
- Because of this, forces always come in pairs that are opposite in direction but equal in magnitude

Translational Equilibrium

- An object is in equilibrium if it is not accelerating (this is true for all inertial reference frames since acceleration is the same regardless of reference frame)
- Therefore the object is not subject to any net force
- The object will remain at rest of move at constant velocity

Lecture 17, Oct 21, 2021

Newtons Laws From Momentum Conservation

- Newton's first law (isolated objects stay at rest/in motion) is a result of momentum being conserved in an isolated system
- Newton's second law ($F = ma$, definition of force) is a result of differentiating momentum $\frac{\mathrm{d}p}{\mathrm{d}t}$
- Newton's third law (every force has an equal and opposite reaction) is a result of differentiating $\Delta p_1 = -\Delta p_2$

Superposition of Forces

- Forces can be superimposed; the result of several forces on an object is an acceleration equal to the sum of the accelerations caused by the individual accelerations

Springs and Tension

- Springs extend to generate forces and pull loads into equilibrium
- The force generated by a spring as the result of a displacement is proportional to the magnitude of displacement (Hooke's law)
- This is only valid for a limited range of extensions and contractions

Impulse

- $\vec{a} = \frac{\Delta\vec{v}}{\Delta t} \implies m\vec{a} = m\frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}$
- We define $\Delta\vec{p} = \vec{J}$, the *impulse* (change in momentum)
- Since $\sum \vec{F} = m\vec{a}$, so $\vec{J} = \Delta t \sum \vec{F}$
- If force varies over time then $\vec{J} = \int_{t_i}^{t_f} \sum \vec{F}(t)\,\mathrm{d}t$
- As area under the acceleration curve is change in velocity, area under the force curve is impulse

Example: Tennis Racket Launching Ball

- Suppose a tennis racket hits a ball (0.20kg) and immediately after the collision the ball has an acceleration of $9g$ upwards, how much force needs to be applied by the racket?

- $\sum F = F^G_{Eb} + F^C_{rb} = m_b a \implies F^C_{rb} - m_b g = 9m_b g \implies F^C_{rb} = 10m_b g$
  * Note superscripts are used to indicate type of force (**G**ravity, **C**ontact), and subscripts are used to denote the objects (**E**nvironment on ball, **r**acket on ball)
- Substitute $m_b = 0.20\text{kg} \implies F^C_{rb} = 20\text{N}$

Lecture 18, Oct 25, 2021

Systems of Interacting Objects

- If two masses are connected by a spring and you push on one of them, what happens to the system?
- What happens to the individual blocks is hard to determine, but the acceleration of the centre of mass is equal to the force over the total mass, because all internal forces cancel out
- The centre of mass of a system behaves like a rigid body so it can be used as a simplification
- The impulse is then equal to $\vec{J} = m_{tot}\Delta\vec{v}_{cm}$

Terminal Velocity/Free Fall

- Objects in free fall are slowed by air resistance; at low speeds this drag force is proportional to speed, and at higher velocities the airflow becomes turbulent and the drag force becomes quadratic
- Since drag force increases as velocity increases, at some point the force of gravity balances with the drag force and the object stops accelerating
- $v_T = \frac{mg}{b}$ for laminar flow or $v_T = \sqrt{\frac{mg}{c}}$ for turbulent flow
- In the case of laminar flow, when approaching terminal velocity, $F = m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv = v_T b - bv \implies \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{b}{m}(v - v_T)$ since $v_T = \frac{mg}{b}$; $\frac{b}{m}$ is a property of the falling object, so define $\tau = \frac{m}{b} \implies \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\tau}(v - v_T)$
  - τ is a time constant characteristic of how fast the object approaches v_T
- This is a differential equation and ends up being a decaying exponential

Lecture 19, Oct 27, 2021

Work

- Definition: Work is change of the energy of a system ΔE as a result of external forces
- Forces can only do work if their point of application is displaced, e.g. pushing on a wall does no work
- Work can result in changes in all forms of energy
- Since energy in a closed system is conserved, work only applies when energy enters or leaves a closed system: $E_f = E_i + W$
- Work is positive if the point of application moves in the same direction as the applied force and negative if it moves in the opposite direction
- Consider two blocks with a spring between them; if the system is considered to be the spring and the blocks, then the system is closed and there is no work since no change in energy happens; if the system is considered to be just the spring, the blocks would do positive work on it
- When picking the system, avoid picking a system that would have friction acting on the boundary; since friction creates thermal energy, which goes into both the system and what's outside the system, it's difficult to tell how much energy is going into the system; furthermore friction is not acting on a single point, so Δx_f cannot be determined

Lecture 20, Oct 28, 2021

Calculating Work

- For a single particle not subject to potential energy change, $\Delta E = W$ and $\Delta E = \Delta K \implies W = \Delta E$
- $v_f = v_i + a\Delta t$ and $\Delta x = v_i\Delta t + \frac{1}{2}a\Delta t^2$, so
$$\begin{aligned} W &= \Delta K \\ &= K_f - K_i \\ &= \frac{1}{2}m(v_f^2 - v_i^2) \\ &= \frac{1}{2}m((v_i + a\Delta t)^2 - v_i^2) \\ &= \frac{1}{2}m(2v_i a\Delta t + a^2\Delta t^2) \\ &= ma(v_i\Delta t + \frac{1}{2}a\Delta t^2) \\ &= ma\Delta x \\ &= F\Delta x \end{aligned}$$
- So work is equal to the product of the force and the displacement of the point the force acts on
- For more than one force $W = \Delta x \sum F_x$
- A closed system has $W = 0$
- There is a parallel between work and impulse; work is to energy as impulse is to momentum
- Problems can be solved in two ways: by considering everything in the system and making it a closed system (chapter 5, energy conservation based approach), or only include some parts and make some forces external (work based approach)

Multiparticle Systems

- For a multiparticle system some work goes into internal energy, so while ΔK is the applied force times to the displacement of the centre of mass, this is not quite W
- However, all work ends up somewhere, so $W_{env} = -W_{sys}$ and we can use the work done by the environment to find the work done on the system
- If there are multiple forces, the force displacement of each point might not all be equal, so $W = \sum F_i\Delta x_i$

Changing Forces

- If the force is varying then an integral can be used

Lecture 21, Nov 1, 2021

Work Done on Springs

- The reaction force of a spring is $k(x - x_0)$ by Hooke's law and if we set the relaxed length to 0, we get $F = -kx$; since this force is equal and opposite to the applied force, the applied force is kx
- The work done by the hand compressing the spring is $\int_0^{x_b} kx\,\mathrm{d}x = \frac{1}{2}kx_b^2$
- The potential energy of the spring is a quadratic function
- Springs are an example of *stable* equilibrium; at $x = x_0$, the spring will actively resist any force that pushes it away from equilibrium
  - *Unstable equilibrium* is like a ball on a hill; if it is pushed even slightly, it will tend towards some other state instead of returning to the same state

Power

- Power is the rate at which energy changes, $\frac{\mathrm{d}E}{\mathrm{d}t}$ or $\frac{\Delta E}{\Delta t}$ if the rate is constant; alternatively $P = Fv$
- Power is measured in J/s = W

Two-Dimensional Motion

- In two dimensions quantities are represented as two-dimensional vectors $\begin{bmatrix} x \\ y \end{bmatrix}$
- The vectors can be in any coordinate system, e.g. the axes can be rotated or stretched

Lecture 22, Nov 3, 2021

Two Dimensional Kinematics

- Velocity and acceleration in 2 dimensions can be decomposed into components
- The velocity $\vec{v}$ is always tangent to the trajectory
- Acceleration $\vec{a}$ can be decomposed into two components, one tangent to the trajectory (same direction as $\vec{v}$) which only accelerates the object, and one perpendicular to the trajectory (orthogonal to $\vec{v}$) which only changes the object's direction
  - This is true instantaneously; over a time interval, as the perpendicular component of acceleration changes the direction of velocity, more and more of the acceleration will go to increasing the velocity rather than changing its direction
- In multiple dimensions position $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, $\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t) = \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}z}{\mathrm{d}t} \end{bmatrix}$ and so on for acceleration
  - The speed is the magnitude of $\vec{v}$
- When we differentiate/integrate to get from one to the other, we can do so for each component separately
- We typically want to find the simplest coordinate system, usually one that has no motion in one or more dimensions, in orders simplify the problem

Projectile Motion

- The special case of constant acceleration where $\vec{a} = -g\hat{\jmath}$
- Horizontal component of velocity is constant (ignoring air friction)
- Since the slope of the curve is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v_y}{v_x}$
- $\vec{r} = \frac{1}{2}\vec{a}t^2 + \vec{v}t \implies \begin{cases} x = v_x t = v_0 \cos\theta t \\ y = -\frac{1}{2}at^2 + v_y t = -\frac{1}{2}gt^2 + v_0 \sin\theta t \end{cases}$; eliminate t by substituting $t = \frac{x}{v_0 \cos\theta}$ to get $y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}$
- To find the range, we solve for when $y = 0 \implies \frac{gx^2}{2v_0^2\cos^2\theta} = x\tan\theta \implies x = \frac{v_0^2}{g}\sin(2\theta)$ (use double angle identity $2\sin\theta\cos\theta = \sin(2\theta)$)
  - In order to maximize range $\sin(2\theta) = 1 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4} = 45°$
- The max flight time only depends on v_y, and occurs when $y = 0 \implies v_0 \sin\theta t = \frac{1}{2}gt^2 \implies t = \frac{2v_0\sin\theta}{g}$, so the object should be thrown straight up

Decomposing Forces

- Example: Pulling someone on a swing; the force needed grows larger as the angle of the swing increases
- We can decompose the force being applied by the swing pivot into two components, one perpendicular the ground and one parallel to the ground
- As you pull the person back, the angle increases and the lateral component of the swing pivot force grows, which needs to be balanced out by the applied force
  - The vertical component of the swing pivot force is balanced by gravity; the diagonal force of the pivot has a vertical component to counteract gravity, and as a side effect this produces a horizontal component that must be balanced out by the applied force

Lecture 23, Nov 4, 2021

Static Friction

- Friction of all kinds always opposes relative motion between two surfaces
- Static friction is a non-dissipative force; since the object does not move, no thermal energy is produced
- Static friction is always less than the net applied force
- Max static friction is proportional to the normal force by a friction coefficient $F_s = \mu_s F_n$; once this amount is exceeded the object will start moving and kinetic friction comes into effect
- Independent of contact area and velocity

Kinetic Friction

- (Kinetic) friction in general is a dissipative force that takes kinetic and/or potential energy to thermal energy; therefore it is irreversible
- Once the object starts sliding, static friction turns into kinetic friction
- Also proportional to the normal force by another coefficient μ_k, which is always less than μ_s
  - If the kinetic friction coefficient is greater than or equal to the static friction coefficient, then the object would stop sliding as soon as it starts sliding
- Once the object starts moving the force required it to keep moving decreases as the type of friction switches from static to kinetic

Decomposition of Friction

- Consider a block resting on a surface; the force of friction acts parallel to the surface and holds the block in place, and opposes the component of gravity that parallel and down the surface
- With these inclined plane problems, forces such as gravity should be broken down into a component parallel to the plane and another one normal to the plane; the normal forces are always in balanced, and the parallel forces are opposed by friction
  - The component of gravity down the plane is $\sin\theta$ and the component normal to the plane is $\cos\theta$; thus at $\theta = 0$ we have a flat plane, and all the force is normal to the plane, and as θ increases, the component of gravity down the ramp increases as $\sin\theta$ increases

Friction Example

- Example: Person jumping onto a slider with $m_p = m_s$, and friction between the person and the block but not between the block and the floor
  - To find the final velocity of the person-block system after the person stops sliding on the block, we can use conservation of momentum $m_p v_{pi} = (m_p + m_b)v_f \implies v_f = \dfrac{v_{pi}}{2}$
  - Kinetic energy follows now that we have the final velocity, so now we can find the change in kinetic energy

- To find the distance that the person slides on the block, we can use $\Delta E_{th} = -\Delta K$ and $F_k = \mu_k m_p g$ so $d = \dfrac{\Delta E_{th}}{\mu_k m_p g}$

Lecture 24, Nov 15, 2021

Work in Multiple Dimensions

- In multiple dimensions power is generalized from $P = Fv$ to $P = \vec{F} \cdot \vec{v}$
- Power is the rate of change of mechanical energy
- In one dimension $K = \frac{1}{2}mv^2 \implies \frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}v^2 = mv\frac{\mathrm{d}v}{\mathrm{d}t} = mva = Fv$
- In multiple dimensions $K = \frac{1}{2}m\|\vec{v}\|^2$ where $\|\vec{v}\|^2 = v_x^2 + v_y^2 + v_z^2$, so $\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{v}\|^2 = \frac{1}{2}m(2v_xa_x + 2v_ya_y + 2v_za_z) = F_xv_x + F_yv_y + F_zv_z = \vec{F} \cdot \vec{v}$
- Using the dot product $\vec{F} \cdot \vec{v} = F_xv_x + F_yv_y + F_zv_z = \left\|\vec{F}\right\|\|\vec{v}\|\cos\theta$, which computes the velocity of the point of application in the direction of the applied force
  - Example: The earth orbiting around the sun does no work, because the gravitational force pulling it inwards is perpendicular to the displacement of the Earth
- Generalizing this to work, $\mathrm{d}W = P\,\mathrm{d}t = \vec{F} \cdot \Delta\mathrm{d}\vec{r}$ and taking the integral we obtain work as a line integral: $W = \int \vec{F} \cdot \mathrm{d}\vec{r}$
  - To actually compute this integral we need to parameterize $\vec{r}(t)$, and turn $\mathrm{d}\vec{r}$ into $\begin{bmatrix}\mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z\end{bmatrix}$, each of which expressed in terms of $\mathrm{d}t$, so the integral can now be computed
  - $\int \vec{F} \cdot \mathrm{d}\vec{r} = \int \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t$ and $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ is just the velocity

Force Created by Potential Energy

- Recall that systems always accelerate towards lower potential every
- Since $\mathrm{d}K = W\,\mathrm{d}t = F\,\mathrm{d}x$, thus $\mathrm{d}U = -F\,\mathrm{d}x$ and $F = -\frac{\mathrm{d}U}{\mathrm{d}x}$
- So if we know the change in potential energy then we know the force
- In multiple dimensions $F_x = -\frac{\partial U}{\partial x}, F_y = -\frac{\partial U}{\partial y}$, etc
- If we write force as a vector then $\vec{F} = -\frac{\partial U}{\partial x}\hat{\imath} + -\frac{\partial U}{\partial y}\hat{\jmath}$, which is the *gradient* of U, $\vec{F} = -\vec{\nabla}U$, thus force is the gradient of potential
  - e.g. gravitational potential $U(x, y) = mgy \implies \vec{F} = \begin{bmatrix}0 \\ -mg\end{bmatrix}$, or a spring $U(x, y) = \frac{1}{2}kx^2 \implies \vec{F} = \begin{bmatrix}-kx \\ 0\end{bmatrix}$
- Forces derived from the gradient of a potential are *conservative*; $U + K$ is always conserved
- With a conservative force the work done is independent of the path taken, by the fundamental theorem of calculus for line integrals

Lecture 25, Nov 17, 2021

Circular Motion

- There are two kind of circular motion: rotations, where the axis is within the object, and revolutions, where the axis is external and out side the object
- With circular motion we can change our coordinate system to use directions that are more convenient

- Use polar coordinates (r, θ) to describe coordinates instead of Cartesian coordinates allows us to only consider θ if the radius remains constant
    - $x = r\cos\theta$ and $y = r\sin\theta$
- Objects in circular motion experience acceleration even when the rate of rotation and radius are constant; we have to fall back to $\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ and $\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}$
- T is the period (time for one revolution), ω is the angular frequency in radians per second (not revolutions per second), $\omega = \frac{2\pi}{T}$
- Define $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(t) - \theta$ is the rotational analogue of position and ω is the rotational analogue of velocity
- Define $\frac{\mathrm{d}\omega}{\mathrm{d}t} = \alpha(t)$, the rotational analogue of acceleration

Uniform Circular Motion

- Motion around a circle with constant radius at a constant speed
- Speed is circumference over time, so $\|\vec{v}\| = \frac{2\pi r}{T} = \frac{2\pi r}{\frac{2\pi}{\omega}} = \omega r$, i.e. speed is angular speed times radius
- The x and y positions over time follow a cosine and sine curve respectively since $\vec{r} = r\begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix}$
    - Note angle $\theta = \omega t$ since constant angular velocity
- $\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}r\cos\theta\\ r\sin\theta\end{bmatrix} = r\frac{\mathrm{d}\theta}{\mathrm{d}t}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$
- $\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$

$$
\begin{aligned}
&= r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\begin{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t} - \cos\theta\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} - \sin\theta\end{bmatrix}\\
&= r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\\
&= r\alpha\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} - r\omega^2\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}
\end{aligned}
$$

  - From these directions we can break $\vec{\alpha}$ into two components and define $\vec{a}(t) = \vec{a}_t + \vec{a}_r$ where $\vec{a}_t = r\alpha\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$, which is parallel to $\vec{v}$, and $\vec{a}_r = -r\omega^2\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$, which is opposite to the direction of $\vec{r}$
  - $\vec{a}_r$ is an inwardly directed acceleration that changes the direction of $\vec{v}$ to keep it in a circle (but not the magnitude); it is referred to as the *centripetal acceleration*
  - $\vec{a}_t$ changes only the magnitude of $\vec{v}$ and speeds the object up or slows it down in circular motion
  - These two directional accelerations are always orthogonal to each other
  - The magnitude of $\|\vec{a}_r\| = r\omega^2 = \frac{v^2}{r}$, and $\|\vec{a}_t\| = r\alpha$
- We can make our lives even easier by making our axes move as well, with the tangential axis being always tangent to the circle of motion and the radial axis being aligned with the radius, and the z axis going out of plane; with this configuration we can do things more naturally without breaking them up into components
- The advantage of using ω instead of $\vec{v}$ is that in a spinning object, every point has the same ω but not the same $\vec{v}$

Angular Kinematics

- The kinematic equations have rotational analogues
- For a constant α (angular acceleration):

- $\alpha(t) = \alpha_0$
- $\omega(t) = \alpha_0 t + \omega_0$
- $\theta(t) = \frac{1}{2}\alpha_0 t + \omega_0 t + \theta_0$
- And all the other ones # Lecture 26, Nov 22, 2021

Forces in Circular Motion

- Since there is an acceleration towards the centre of motion, there must be a force $\frac{mv^2}{r}$ pushing the object towards the centre (centripetal force)
  - This means that as radius increases the force needed to pull the object into circular motion decreases (at the extreme for an object moving in a straight line the radius is effectively infinite)
    - Smaller radii means more change in direction per unit time (for the same ω) so the vector change in velocity is greater
  - As velocity increases the centripetal force also increases

Banked Curves

- On a banked curve, the normal force now has a component towards the middle of the track and the vector is tilted
- The component towards the middle becomes the centripetal force
- The vertical component of normal force needs to be large enough to balance out gravity
- The masses cancel out and $\tan\theta = \frac{v^2}{gr}$

Rotational Inertia

- Objects are harder to rotate the further you are from their centre of mass, so both radius and mass affects inertia for rotation
- For example you hold a hammer from the end, far away from its centre of mass, so you can store more energy and momentum in it for the strike
- The kinetic energy of a point mass that's revolving is $K = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}(mr^2)\omega^2$
  - Comparing this to $\frac{1}{2}mv^2$ we see that mr^2 is the equivalent of m, so we write it as $I = mr^2$ and the angular kinetic energy $K_{rot} = \frac{1}{2}I\omega^2$
  - I is the *rotational inertia*
- Rotational momentum works in the same way; if an object of mass m moves at speed v in a straight line and then strikes another object with the same mass, all the momentum will be transferred, so the second object now has momentum $mv = mr\omega \implies rmv = I\omega$
  - Define $I\omega$ as the *angular momentum*
  - Since $\omega = \frac{v}{r}$ the further along the radius of object 2 that object 1 hits, the easier it is to set object 2 into rotational motion, so we say that objects have more rotational momentum if their radius is further
  - Note since $I\omega = rmv$ an object moving linearly has angular momentum if we can define r; these objects have angular momentum because they are able to set objects into rotational motion
  - For a particle moving linearly we define r as the perpendicular distance from the line of action of the particle and the rotational axis, or the cross product of the object's velocity and the radius vector
- Angular momentum is a conserved quantity

Computing Rotational Energy

- If we apply $K = \frac{1}{2}mv^2$ to all particles, then for a rotating object the kinetic energy of the whole object is $K = \frac{1}{2}I\omega^2$ where $I = \int r^2 \, \mathrm{d}m$
  - Since every point in the object has the same ω, this is a constant
  - Each piece of the object has energy $\frac{1}{2}v^2 \, \mathrm{d}m = \frac{1}{2}I\omega^2$ where the I for this piece is $r^2 \, \mathrm{d}m$
  - Since ω is a constant we can pull it out of the integral to get $\frac{1}{2}\omega^2 \int r^2 \, \mathrm{d}m$
  - Caveat: The object cannot deform since that would change the r
  - To compute this integral we say $\mathrm{d}m = \rho \, \mathrm{d}V = \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$ so for a 3D object this becomes a triple integral $\iiint r^2 \rho(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$
    - Generalizes to other dimensions
- Example: Linear thin rod (effectively 1 dimensional) $I = \int r^2 \, \mathrm{d}m = \rho \int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}\rho(b^3 - a^3)$, and if the object is L long and attached to the axis this is equal to $\frac{1}{3}\rho L^3$
  - Note: Since density $\rho = \frac{m}{L}$, this is also equal to $\frac{1}{3}mL^2$

Lecture 27, Nov 24, 2021

Parallel Axis Theorem

- The moment of inertia about some axis $I = I_{cm} + Md^2$ where d is the distance from this other axis and the centre of mass axis
  - This only works when the axis of I and I_{cm} are parallel
- As a consequence of this, $K = \frac{1}{2}I\omega^2 = \frac{1}{2}(I_{cm} + Md^2)\omega^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Md^2\omega^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv^2$
  - There is a component of kinetic energy from pure rotation about the centre of mass and another from the translational kinetic energy

Torque and Angular Momentum Change

- Change in angular speed requires a force that acts in the tangential direction: torque
- Since any applied force only speeds up an object in rotation if it acts in the tangential direction, torque is computed by $Fr \sin \phi$ where ϕ is the angle between the radius and force vectors
  - We can also look at the line of action and perpendicular lever arm $r_\perp$
  - We can also look at it as the vector cross product: $\vec{\tau} = \vec{r} \times \vec{F}$
    - The magnitude of this vector is the magnitude of torque, and it points in the direction of the axis of rotation

Lecture 28, Nov 25, 2021

Free Rotation

- When an object is allowed to rotate and translate freely, it will rotate around its centre of mass
- The centre of mass undergoes essentially no rotation and is in translational motion only; the translational and rotational movements are essentially uncoupled
- This means we can decompose the kinetic energy into two parts: the translational kinetic energy of the centre of mass, and the rotational kinetic energy of the rest of the object

Extended Free Body Diagrams

- In an EFBD both rotation and translation are considered
- The rotation and translation are decoupled by separating it into the centre of mass translation and rotation about the centre of mass
- Show both the torques (forces with a lever arm about the rotational axis) and other forces that pass through the centre of mass
  - In a regular FBD all forces are shown to act on the centre of mass, but in an EFBD the forces have lever arms
  - The EFBD can be used to calculate both translational acceleration and angular acceleration

Vector Nature of Rotation

- Rotations in 3D can be described by a vector where the magnitude of that vector is the rotational angle/speed/acceleration and the direction is the axis of rotation
- θ, ω and α essentially only have two dimensions, counterclockwise or clockwise
- By convention counterclockwise rotation is positive and clockwise motion is negative
- The direction of the rotation vector (in or out of the plane) determines the direction of rotation; use the right hand rule (curl your right hand's fingers in the direction of rotation, then the thumb points in the direction of the rotation vector)
- This vector describes the direction, magnitude, and axis of rotation so it completely describes the rotation
- Angular momentum $\vec{L}$ can now be considered as the cross product $\vec{L} = \vec{r} \times \vec{p}$, just like torque $\vec{\tau} = \vec{r} \times \vec{F}$
  - Like torque we can consider either the angle between the lever arm and the linear momentum, or the perpendicular lever arm
  - Since the cross product points out of the plane, the angular momentum vector is pointed in the direction of the axis of rotation

Relationship Between $\vec{L} = I\vec{\omega}$ and $\vec{L} = \vec{r} \times \vec{p}$

- Consider a point mass in revolutionary motion with radius $\vec{r}$ and velocity $\vec{v}$
- We know $\vec{v} = \|\vec{r}\|\vec{\omega}$
- Let $\hat{r}$ be the unit vector in the radial direction and $\hat{t}$ be the unit vector in the tangential direction, and $\hat{z}$ be the unit vector pointing out of the plane of rotation
- $\vec{L} = \vec{r} \times \vec{p} = r\hat{r} \times mv\hat{t} = rmv\hat{r} \times \hat{t} = rmv\hat{z} = r^2 m\omega\hat{z} = I\vec{\omega}$
- For an extended object we can sum up all the small pieces of the mass for $\vec{L} = \sum I_i\vec{\omega} = I\omega\hat{z}$

Lecture 29, Dec 1, 2021

Simple Harmonic Motion

- Simplest example of an oscillator is a mass on a spring
- The restoring force is $-kx$ so $F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \implies \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0$
  - The solution to this DE is a sinusoid $A\cos(\omega t + \phi)$
  - Substituting this back in we get $\omega = \sqrt{\frac{k}{m}}$
- Simple harmonic motion is like looking at uniform circular motion in one axis
- When the system oscillates, there is a constant conversion between kinetic and potential energy
  - At the extremes all the energy is contained in the spring as potential energy
  - When the system passes through equilibrium all the energy is kinetic
  - In-between there's a mix of the two
- Simple harmonic oscillators have the same period regardless of amplitude; no matter how far you pull the spring when you start it, it will still take the same time to complete a full cycle (ω does not depend on A)

- $\begin{cases} x(t) = A\cos(\omega t + \phi) \\ v(t) = -A\omega\sin(\omega t + \phi) \\ a(t) = -A\omega^2\cos(\omega t + \phi \end{cases}$
  - Maximum displacement is A
  - Maximum velocity is $A\omega$
  - Maximum acceleration is $A\omega^2$

Lecture 30, Dec 2, 2021

Approximation of Linear Restoring Forces

- Simple harmonic motion occurs whenever there is a linear restoring force
- Usually restoring forces can be approximated linearly even when they're not (for small displacements)
- This means that there is usually simple harmonic motion occurring whenever there is a point of stable equilibrium (forces attempt to restore the object to equilibrium instead of pushing it away or doing nothing when the object goes away from equilibrium)
- If $F_x = -kx$ then $U = -\int F_x \,\mathrm{d}x = \frac{kx^2}{2} + U(0)$
  - $k = \frac{\mathrm{d}^2U}{\mathrm{d}x^2}$ is the curvature of the potential, and we can derive this approximate k even when the restoring force is not truly linear
  - This can be done for any potential but the motion is only truly harmonic for k very close to equilibrium points
- A pendulum is such an example; the restoring torque is not truly linear but for small angles $\sin x \approx x$
  - Potential is parabolic around the minimum for true harmonic motion
  - Since the pendulum is constrained to circular motion, its potential (w.r.t. angle) is sinusoidal
  - In a simple pendulum there is negligible mass in the string, the pendulum has a force mg pointing straight down from gravity, acting on the mass
  - The radial component is $mg\cos\theta$ and the tangential component is $mg\sin\theta$
    * The radial component is opposed by the tension in the string but the tangential component is not
    * Sum of the forces in the radial direction is $mg\cos\theta - T = ma_r$
    * $a_r = -a_c = r\omega^2$ pointing in the $-\hat{r}$ direction towards the centre
    * Since the pendulum doesn't move radially this means that $mg\cos\theta = T$
    * Forces in the tangent direction is $mg\sin\theta$ without a counterbalancing force, so the tangential acceleration is $r\alpha = la_\theta$
    * $-mg\sin\theta = ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \implies \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mg}{ml}\sin\theta = 0$ is the equation of motion for the simple pendulum
      - This would be the same equation as simple harmonic motion if the $\sin\theta$ was instead θ
    * Behaviour is non harmonic if $\sin\theta$ is not close to θ, when $\frac{\theta^3}{3!}$ is not negligible
      - Expanding this out the contribution is $\frac{\frac{\theta^3}{3!}}{\theta}$, so we want $\frac{\theta^2}{6} \ll 1$
  - In a physical pendulum in which the mass of the rod cannot be ignored, we can use I instead of m and now the gravitational force acts on the centre of mass of the entire rod-mass system
    * The torque applied is $mgl\sin\theta$ where l is the distance of the centre of mass of the system from the pivot
    * For the physical pendulum $\sum \tau = -mgl\sin\theta = I\alpha_\theta = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mgl\sin\theta}{I} = 0$
      - If $I = ml^2$ then this would be the same as the simple pendulum equation
  - Compare to simple harmonic motion $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$
  - Even for a spring, $F = -kx$ is only an approximation

- For any arbitrary (differentiable) potential we can always approximate its behaviour at minima using parabolas, which lead to simple harmonic motion

Lecture 31, Dec 6, 2021

Simple Harmonic Oscillator Energy

- Recall $a(t) = -A\omega^2\cos(\omega t + \phi) \implies F_x = -mA\omega^2\cos(\omega t + \phi) = -m\omega^2 x(t)$
- We can integrate this force to find the work done: $W = \int_{x_0}^{x} -m\omega^2 x \, \mathrm{d}x = \frac{1}{2}m\omega^2(x_0^2 - x^2)$
- $W = \Delta K \implies \Delta K = \frac{1}{2}m\omega^2(x_0^2 - x^2)$
- If we make the oscillator a closed system then $\Delta U = -\Delta K = \frac{1}{2}m\omega^2(x^2 - x_0^2)$
  - Since potential at x_0 is arbitrary we can set this to 0 so $\Delta U = \frac{1}{2}m\omega^2 x^2$
- The total energy is then $E = U + K = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi) = \frac{1}{2}m\omega^2A^2$
  - The energy of a simple harmonic oscillator is constant and it only trades potential and kinetic energy back and forth
  - The energy is proportional to the square of the amplitude

Torsional Oscillators

- A disk suspended by a twisting fibre creates a torsional oscillator in simple harmonic motion
  - $\tau = I\alpha$
  - For small angular displacements $\tau = -\kappa(\omega - \omega_0)$ where κ is the angular equivalent of k
  - The differential equation is then $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{\kappa}{I}\theta$ which is the same as the one for translational simple harmonic motion, just with the translational terms substituted by rotational ones
- The equations of motion are identical to translational simple harmonic motion $\theta = \theta_{max}\cos(\omega t + \phi)$ and $\omega = \sqrt{\frac{\kappa}{I}}$

Examples of Oscillating Systems

- All simple harmonic oscillators obey $\frac{\mathrm{d}^x t}{\mathrm{d}+^x}\omega^2 x = 0$
- Examples:
  1. Mass on a spring: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0 \implies \omega^2 = \frac{k}{m}$
  2. Torsional oscillator: $I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \kappa\theta = 0 \implies \omega^2 = \frac{\kappa}{I}$
  3. Pendulum: $ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl\sin\theta \approx ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl\theta = 0 \implies \omega^2 = \frac{g}{l}$
  4. Floating object bobbing in water: $m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + g(\rho A y) = 0 \implies \omega^2 = \frac{g\rho A}{m}$
     - A buoyant object will float in the water at some neutral point, and if pushed past this neutral point then $F_y \propto y$
  5. Capacitor-inductor circuit $L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{Q}{C} = 0 \implies \omega^2 = \frac{1}{LC}$
     - Voltage drop across capacitor: $\frac{Q}{C}$
     - Voltage drop across inductor: $L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}$

Damped Oscillations

- In reality there is always some friction present, causing the oscillator to lose energy and thus amplitude
- The loss in amplitude is called *damping*
- Often the damping is caused by viscous friction, with the damping force opposing and proportional to velocity $F = -b\vec{v}$
- Therefore $F_x = -(kx + bv_x) = ma_x \implies m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$
- The solution is given by $x(t) = Ae^{-\frac{bt}{2m}}\cos(\omega_d t + \phi)$ where $\omega_d = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
  - A damped harmonic oscillator oscillates slower than the equivalent undamped oscillator
  - Let the damping time constant $\gamma = \frac{b}{m}$, with units of time
  - The amplitude can be expressed as $x_{max} = Ae^{\frac{-\gamma t}{2}}$
  - The energy remaining is then $E(t) = \frac{1}{2}m\omega^2 x_{max}^2 = \left(\frac{1}{2}\omega^2 A^2\right)e^{-\gamma t}$
- As b increases the oscillator decays faster, and when $b > 2m\omega$ the system is overdamped and there are no oscillations at all

Q Factor

- Q is the quality factor of an oscillator and measures the rate of decay
  - Differs from ω in the sense that Q measures the number of oscillations
- $Q \equiv \frac{\omega}{\gamma} = \frac{2\pi}{\gamma T}$
- If $Q = 2\pi$ then the energy falls to $e^{-1} = 37\%$ of its original energy in a single cycle
- A good bell has $Q = 100$, electronic circuits have $Q = 10^6$, quantum systems an have $Q = 10^9$