Lecture 9, Sep 29, 2021

The Derivative

- Define the derivative $f'(a) \equiv \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ if it exists, where $a \in$ domain of $f(x)$
- Example: $f(x) = x^3$, $f'(1)$
  - $$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^3 - 1^2}{h} \\ &= \lim_{h\to 0} \frac{1+3h+3h^2+h^3-1}{h} \\ &= \lim_{h\to 0} \left[h^2+3h+3\right] \\ &= 3 \text{ by the Polynomial LT} \end{aligned}$$
  - h disappears here; it is a dummy variable and the derivative does not depend on it
- With the rigorous definition of the derivative, we can now define rigorously the slope of a tangent, velocity at an instant, etc

Derivative as a Function

- $f'(a)$, the derivative of a, is just a number
- We can define a new function $f'(x) \equiv \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$; now both x and h are variables
- Just like before, h will still disappear when the limit is evaluated, so $f'(x)$ is only a function of x
- In evaluating the limit, we treat x as if it were a constant
- If $f'(a)$ exists at a, then we define $f(x)$ to be *differentiable* at a
  - If $f(x)$ is differentiable at all $x \in$ domain of $f(x)$, then we define $f(x)$ to be a *differentiable function*
- Example: Prove $f(x) = x^n \implies f'(x) = nx^{n-1}$ for positive integer n
  - Note the binomial theorem: $$\begin{aligned} &(a+b)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots \end{aligned}$$
  - $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\ &= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right] \\ &= nx^{n-1} \end{aligned}$$