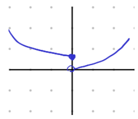
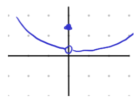
Lecture 8, Sep 27, 2021

Continuity & Discontinuity

- The formal definition of continuity at a point is $f(x)$ is continuous at c iff $\lim_{x \to c} f(x) = f(c)$
- Types of discontinuity:
  1. Jump discontinuity: 
  2. Removable discontinuity: 
  3. $\lim_{x \to c}$ DNE or $f(c)$ DNE

Continuity Theorems

- Many important theorems hold true only for continuous functions, so it is important to be able to prove that functions are continuous
  - e.g. Continuity implies integrability; if $f(x)$ is continuous on $[a, b]$ then $\int_a^b f(x)\,\mathrm{d}x$ is guaranteed to exist (note: it might not be able to be expressed using already defined mathematical operations, but it must exist)
- There are continuity theorems just like limit theorems:
  1. Additivity Continuity Theorem: $f(x)$ and $g(x)$ continuous at $c \implies f(x) + g(x)$ continuous at c
     - Proofs are trivial using the corresponding limit theorems
  2. Product Continuity Theorem: $f(x)$ and $g(x)$ continuous at $c \implies f(x)g(x)$ continuous at c
  3. ...
- Composite Function Continuity Theorem: Given $g(x)$ is continuous at a and $f(x)$ is continuous at the number $g(a)$, then $f(g(x))$ is continuous
  - Using this we can test continuity for nested expressions, e.g. $f(x) = \tan\left(\frac{\pi x^2}{4}\right)$ at $x = 1$

One-Sided Continuity

- Just like limits, continuity can also be one-handed
- $f(x)$ is continuous on the left at c iff $\lim_{x \to c^-} f(x) = f(c)$; $f(x)$ is continuous on the right at c iff $\lim_{x \to c^+} f(x) = f(c)$
- e.g. $f(x) = \begin{cases} 1 + x^2 & x \geq 0 \\ x^2 & x < 0 \end{cases}$ is continuous from the right at 0 and discontinuous from the left at 0

Continuity On an Interval

- $f(x)$ is continuous on the open interval (a, b) iff $f(x)$ is continuous for all $x \in (a, b)$
- $f(x)$ is continuous on the closed interval $[a, b]$ iff $f(x)$ is continuous for all $x \in (a, b)$ **and** $f(x)$ is continuous from the right at a **and** $f(x)$ is continuous from the left at b
  - Notice that this doesn't require $f(x)$ to be continuous on both sides at a and b; just one-handed continuity is okay

Intermediate Value Theorem

- Given $f(x)$ continuous on $[a, b]$ and $f(a) < C < f(b)$ or $f(a) > C > f(b)$, there exists some $c \in [a, b]$ such that $f(c) = C$

- The point it that continuous functions don't "skip over" y values; without irrationals and transcendentals the IVT cannot be satisfied so continuity breaks
- Using this we can prove that there exists a number $\sqrt{2}$ such that $\sqrt{2}\sqrt{2} = 2$, which is much harder to do with CORA alone
  - Take $f(x) = x^2$, which is continuous over $[1, 2]$ by the Polynomial CT
  - $f(1) = 1 < 2 < f(2) = 4$ so by the IVT there exists some number $1 < c < 2$ such that $c^2 = 2$
- We can also define transcendentals: $\sin x = 0$ at $x = \pi$, so take $f(x) = \sin x$ and find a point where $\sin x > 0$ and a point where $\sin x < 0$, so this implies that somewhere in this range there exists a c such that $\sin c = 0$
- The IVT is only true because of the existence of irrational numbers, so it could be used to "create" the irrationals; instead of having CORA, we could have started with an Intermediate Value *Axiom* and then proved the Completeness of the Reals *Theorem*, but this is messy because we would need to first define functions

Bonus: Proof of the IVT

- Proof from pages 50-52 of supplement
- Lemma: Given $f(x)$ continuous on $[a, b]$ and $f(a) < 0 < f(b)$, there exists a number $a < c < b$ for which $f(c) = 0$
  - Let S be the set of all numbers $x \in [a, b]$ such that $f(x) < 0$
  - We know that S contains x in some interval $[a, a + \delta)$
    - Because $\lim\limits_{x \to a^+} f(x)$ exists, if we take $\varepsilon = |f(a)|$, then $|f(x) - f(a)| < |f(a)| = -f(a)$ for $a < x < a + \delta$
    - Therefore $|f(x) - f(a)| < |f(a)| \implies f(a) - |f(a)| < f(x) < f(a) + |f(a)| \implies 2f(a) < f(x) < 0$, and since $f(x) < 0$, $x \in S$
  - This means that S is nonempty and bounded above, so by CORA it has a least upper bound c
  - It is not possible for $f(c) > 0$:
    - Suppose $f(c) > 0$
    - There exists $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < f(c)$ (in this case $\varepsilon = f(c)$, which is assumed positive)
    - $|f(x) - f(c)| < f(c) \implies f(c) - f(c) < f(x) < f(c) + f(c) \implies 0 < f(x)$, when $c - \delta < x \le c$ or $c \le x < c + \delta$
    - Since $f(x) > 0$ for $c \in (c - \delta, c)$, these values are not in S and thus are upper bounds of S; but this contradicts the assumption that c is the least upper bound, so $f(c) > 0$ is not possible
  - Similarly $f(c) < 0$ is also not possible, so by process of elimination $f(c) = 0$ (note: $f(c)$ has to exist since $c \in [a, b]$ and $f(x)$ is continuous over that range)
- Now suppose we define $g(x) \equiv f(x) - C$, then $f(a) < C \implies g(a) = f(a) - C < 0$, and $C < f(b) \implies 0 < f(b) - C = g(b)$ and by addition CT $g(x)$ is continuous over $[a, b]$; therefore there exists some c such that $g(c) = 0 \implies f(c) = C$ by the lemma
- Similarly this can be proven for the case where $f(a) > f(b)$