Lecture 7, Sep 23, 2021

Limit Theorems

- Limit theorems let us rigorously prove much more complicated limits, such as polynomials, by breaking it into manageable pieces, each provable with the epsilon-delta definition
- Some limit theorems: (assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ are given; **both limits need to exist**)
  1. Additivity Limit Theorem: $\lim_{x\to c}[f(x) + g(x)] = L + M$
     - Proof:
       * Required $|f(x) + g(x) - L - M| < \varepsilon$ when $0 < |x - c| < \delta$
       * $|f(x) + g(x) - L - M| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|$ (triangle inequality)
       * We are given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, therefore:
         - For some $\varepsilon_f = \frac{\varepsilon}{2}$, there is $0 < |x - c| < \delta_f$ such that $|f(x) - L| < \varepsilon_f = \frac{\varepsilon}{2}$
         - For the same ε_f, there is $0 < |x - c| < \delta_g$ such that $|g(x) - M| < \varepsilon_f = \frac{\varepsilon}{2}$
       * If x is inside both δ_f and δ_g bands, then $|f(x)+g(x)-L-M| \leq |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
       * Therefore by picking $\delta = \min(\delta_f, \delta_g)$, $0 < |x - c| < \delta \implies |f(x) + g(x) - L - M| < \varepsilon$, and the limit is proved!
  2. Product Limit Theorem: $\lim_{x\to c} f(x)g(x) = LM$
     - Proof:
       * From the limits given: For some $\varepsilon_f = \sqrt{\varepsilon} > 0$ there is $0 < |x-c| < \delta_f \implies |(f(x)-L)-0| < \varepsilon_f$ and $0 < |x - c| < \delta_g \implies |(g(x) - M) - 0| < \varepsilon_g$
       * Therefore if $\delta = \min(\delta_f, \delta_g)$ then $0 < |x - c| < \delta \implies |(f(x) - L) - 0||(g(x) - M) - 0| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$, so $\lim_{x\to c}(f(x) - L)(g(x) - M) = 0$
       * $0 = \lim_{x\to c}(f(x) - L)(g(x) - M) = \lim_{x\to c}[f(x)g(x) - f(x)M - g(x)L + LM] = \lim_{x\to c} f(x)g(x) + \lim_{x\to c} -Mf(x) + \lim_{x\to c} -Lg(x) + \lim_{x\to c} LM$ by the additive theorem
       * Therefore $\lim_{x\to c} f(x)g(x) = -\lim_{x\to c} -Mf(x) - \lim_{x\to c} -Lg(x) - \lim_{x\to c} LM = ML + LM - LM = ML$ (by $\lim_{x\to c} cf(x) = c\lim_{x\to c} f(x)$, ~~proof of which is left as an exercise to the reader~~)
  3. Polynomial Limit Theorem: $\lim_{x\to c} P_n(x) = P_n(c)$ for polynomials $P_n(x)$
     - This can be trivially proven using the product and additivity limit theorems
  4. Rational Function Limit Theorem: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
     - Proof not included because I'm tired, see this link.
  5. Root Limit Theorem: $\lim_{x\to c} \sqrt[n]{f(x)} = L^{\frac{1}{n}}$
- Other limit theorems:
  6. Sandwich (aka Squeeze) Limit Theorem: If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ and $f(x) \leq g(x) \leq h(x)$ near but not necessarily at c, then $\lim_{x\to c} g(x) = L$
     - $g(x)$ may be a very complicated function, but we might be able to find simple functions $f(x)$ and $h(x)$ that bound $g(x)$ near c for which we can easily find the limits of

Applying Limit Theorems

- Example: $\lim_{x \to -2} \frac{x^2 - x - 6}{x^2 - 4}$

$$= \lim_{x \to -2} \frac{(x-3)(x+2)}{(x-2)(x+2)}$$
$$= \lim_{x \to -2} \frac{x-3}{x-2}$$
$$= \frac{\lim_{x \to -2} x - 3}{\lim_{x \to -2} x - 2} \text{ (rational function LT)}$$
$$= \frac{5}{4} \text{ (polynomial LT)}$$

- Rigorously logical proofs justify every line by citing an axiom or proven theorem
- Example of using the Sandwich Theorem: $\lim_{x \to 0} x^2 \cos^2 \frac{1}{x^2}$
  - Can't use the product LT because $\lim_{x \to 0} \cos^2 \frac{1}{x^2}$ DNE (the function just oscillates faster and faster)
  - Find bounding functions: $0 \leq \cos^2 \frac{1}{x^2} \leq 1 \implies 0 \leq x^2 \cos^2 \frac{1}{x^2} \leq x^2$
  - Define $f(x) \equiv 0, g(x) \equiv x^2 \cos^2 \frac{1}{x^2}, h(x) \equiv x^2$
  - $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} h(x) = 0$ by the polynomial LT, therefore $\lim_{x \to 0} g(x) = 0$
  - Graph: