Lecture 6, Sep 21, 2021

- The minimum function lets us unambiguously prescribe multiple things about a variable
- Example: Prove $\lim_{x \to 5} x^2 = 25$
  1. $\varepsilon > 0$ is given
  2. Required $|f(x) - L| = |x^2 - 25| < \varepsilon$
  3. When $0 < |x - c| = |x - 5| < \delta$
  4. $|x^2 - 25| = |(x-5)(x+5)| = |x-5||x+5| < \delta|x+5|$
     - Now we need to get rid of $|x+5|$ by specifying an additional feature of δ
     - If we specify $\delta \le 1 \implies |x-5| < \delta \le 1 \implies 4 \le x \le 6 \implies 9 \le x+5 \le 11 \implies |x+5| \le 11$
     - $|x+5| \le 11 \implies \delta|x+5| \le 11\delta \implies |x-5||x+5| < 11\delta$
  5. Now we can take $\delta = \dfrac{\varepsilon}{11} \implies |f(x) - L| = |x-5||x+5| < 11\delta = \varepsilon$
  6. Include constraint $\delta \le 1$: Notice how anything smaller than $\dfrac{\varepsilon}{11}$ works, so we can say $\delta = \min\left(\dfrac{\varepsilon}{11}, 1\right)$

Left and Right-Hand Limits

- Consider $f(x) = \begin{cases} 1 + x^2 & x \ge 0 \\ x^2 & x < 0 \end{cases}$; in this case $\lim_{x \to 0} f(x)$ does not exist
- We can define the *left-hand limit*: $\lim_{x \to 0^-} f(x)$ and the *right-hand limit*: $\lim_{x \to 0^+} f(x)$
- The definition of the left and right hand limits are slight modifications to the normal limit:
  - Right-hand: $\lim_{x \to c^+} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$ such that $\forall c < x < c + \delta, |f(x) - L| < \varepsilon$
  - Left-hand: $\lim_{x \to c^-} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$ such that $\forall c - \delta < x < c, |f(x) - L| < \varepsilon$
  - We can then conclude that $\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$
- Example: Prove $\lim_{x \to 0^+} \sqrt{x} = 0$
  1. $\varepsilon > 0$ is given
  2. Required $|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$
  3. When $0 < x < \delta$
  4. $\sqrt{x} < \sqrt{\delta}$
  5. Take $\delta = \varepsilon^2 \implies |f(x) - L| = \sqrt{x} < \sqrt{\delta} = \varepsilon$

Vertical Asymptotes and Infinite Limits

- e.g. $f(x) = \dfrac{1}{x^4}$ goes to infinity at 0
- The definition of an infinite limit $\lim_{x \to c} f(x) = \infty \iff \forall M > 0, \exists \delta > 0$ such that $\forall 0 < |x - c| < \delta, f(x) > M$
  - Note that this is not an equation since ∞ is not a number!
  - $\lim_{x \to 0} \dfrac{1}{x^4}$ does not exist, but writing $\lim_{x \to c} f(x) = \infty$ is valid and tells us more than just saying it does not exists
  - This is just a shorthand of saying "$f(x)$ increases without limit as x approaches c"
- From this definition we can similarly construct definitions for limits of negative infinity and infinite one-handed limits
- Example: $\lim_{x \to 0} \dfrac{1}{x^4} = \infty$
  - $\dfrac{1}{x^4} > M$ when $0 < |x| < \delta$
  - $|x| < \delta \implies \dfrac{1}{\delta} < \dfrac{1}{|x|} \implies \dfrac{1}{\delta^4} < \dfrac{1}{|x|^4}$
  - Take $\delta = \dfrac{1}{M^{\frac{1}{4}}} \implies \dfrac{1}{x^4} = \dfrac{1}{|x|^4} < \dfrac{1}{\delta^4} = M$