Lecture 5, Sep 20, 2021

Rigorous Definition of the Limit

- Test-definition (a type of implicit definition) for a new number $\lim_{x \to c} f(x)$, given:
  1. c, some particular value of x
  2. $f(x)$, which may be undefined at c, but is defined for all x near c
  3. L, a candidate value for the limit
- Imposed: Some small positive $\varepsilon > 0$; we don't have the exact value and will have to allow for any $\varepsilon > 0$
- Test: Find some $\delta > 0$, such that for all $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$
  - i.e. Find some δ such that all x within the x-band have corresponding values of f that fall in the y-band
- If the test passes, then the limit exists and $\lim_{x \to c} f(x) = L$
- Since $|x - c| > 0$, x is never really equal to c, so we can simplify situations such as $\frac{x}{x}$ when $c = 0$ legitimately

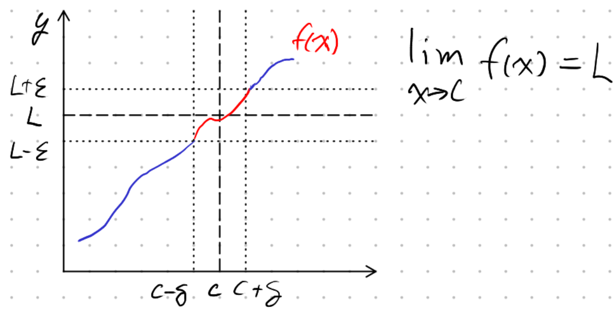

Figure 1: graph illustration

General Process

1. $\varepsilon > 0$ is imposed (it is **given** so we can and have to work with it)
2. Find a set of x values for which $|f(x) - L| < \varepsilon$
   - Example: Prove $\lim_{x \to 0} \frac{10x + 5x^2}{x} = 10$
   - Here $\left| \frac{10x + 5x^2}{x} - 10 \right| = |10 - 5x - 10| = |5x| = 5|x| < \varepsilon$
   - Notice how the x can be cancelled out rigorously now since x is not allowed to be zero
3. Look for a set of x values you will specify by $0 < |x - c| < \delta$
   - Example: $0 < |x - 0| = |x| < \delta$
4. Plug ③ into the left hand side of ② and do algebraic manipulation until you get $|f(x) - L| <$ some expression involving only δ
   - Example: If $5|x| < \varepsilon$ for ② and $|x| < \delta$, then $5|x| < 5\delta$
5. Guess δ in terms of ε and plug back in to get $|f(x) - L| <$ some expression involving ε, and then make the right hand side $< \varepsilon$
   - Example: Choose $\delta = \frac{1}{5}\varepsilon$, substitute into $5|x| < 5 \implies 5|x| < \varepsilon$
   - Now we've found (one of the) δ values for any given ε such that $|f(x) < L| < \varepsilon$ for all $0 < |x - c| < \delta$, so we can conclude $\lim_{x \to c} f(x) = L$
6. Compact: Given $\varepsilon > 0$, take $\delta = \cdots$ then when $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$, therefore $\lim_{x \to c} f(x) = L$

Example

- Prove $\lim_{x \to 0} x^3 = 0$

- $|x^3 - 0| = |x^3| = |x|^3 < \varepsilon$
- $0 < |x - 0| = |x| < \delta$
- $|x|^3 < \delta^3$
- Take $\delta = \sqrt[3]{\varepsilon} \implies |x|^3 < \varepsilon$; QED
- Note the choice of δ is not unique; anything that does the job is fine!