Lecture 4, Sep 17, 2021

Review, Continued

- BAT2 (Basic Algebraic Theorem)
  - Note: $1 + 2 = 6 - 3$ is a *statement* and therefore can be true or false
  - $x + 3 = 7x - 1$ is **not** a statement (not meaningful to ask whether it is true or false); it is a *prescription* for the value of x; $x^2 - 4 = 0$ is a prescription for 2 values of x
  - Similarly $3 < 5$ is a statement (true/false); $x < 4$ is a prescription for a whole set of x values
  - BAT2 is equal to BAT1 except the factor could now be algebraic; since the sign might be hidden inequalities require more work
  - e.g. Prove $f(x) = x^2 + 3$ is decreasing for $x < 0$
    * Pick any $x_1 < x_2 < 0$
    * $x_1 < x_2 \implies x_1^2 > x_1 x_2$ **note the direction change because $x_1 < 0$!**
    * $x_1 < x_2 \implies x_1 x_2 > x_2^2$ same direction change here because $x_2 < 0$
    * See definition of increasing/decreasing functions from Lecture 3 for rest of proof
  - $x^2 - x - 6 > 0 \implies (x - 3)(x + 2) > 0$
    * First subset: $x - 3 > 0$ and $x + 2 > 0$ **for the same x**
    * Second subset: $x - 3 < 0$ and $x + 2 < 0$ **for the same x**
  - Equalities prescribe a number of x values; inequalities prescribe a set of x values (or none at all if the inequality is impossible)
- Absolute value functions $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$
  - Example: $f(x) = |x + 3| = \begin{cases} x + 3 & x + 3 \geq 0 \\ -x - 3 & x + 3 < 0 \end{cases}$
  - Example: What values of x satisfy $|x + 3| = 5$?
    * Possibility 1: $x + 3 > 0 \implies x + 3 = 5$ so $x > -3$ and at the same time $x = 2$, which is possible
    * Possibility 2: $x + 3 < 0 \implies -x - 3 = 5$ so $x < -3$ and $x = -8$, which is also possible
  - Example: $|x + 3| < 5$
    * Possibility 1: $x + 3 \geq 0 \implies x + 3 < 5$ so $x \geq -3$ and $x < 2$, so $-3 \leq x < 2$
    * Possibility 2: $x + 3 < 0 \implies -x - 3 < 5$ so $x < -3$ and $x > -8$, so $-8 < x < -3$
    * Combine the two sets: $-8 < x < 2$
    * -3 is halfway between 2 and -8 with half-width 5 (generalizes to other inequalities of the same form)
      - Example: $|x - c| < \delta$ where c can be positive or negative and $\delta > 0$; $+c$ is the center and δ is the half-width: $c - \delta < x < c + \delta$
  - The set $0 < |x - c| < \delta$ excludes $x = c$; this is important in the context of limits since the function may be undefined at c
    * If $0 < |f(x) - c| < \delta$ says that $f(x)$ can never go above or below $c \pm \delta$, so this bounds its range, useful for limits