Lecture 34, Dec 6, 2021

Variation of Parameters

- A more robust method of finding the particular solution
- The complementary solution is $y_c = C_1 y_1(x) + C_2 y_2(x)$; to begin this method, we try allowing these constants C_1 and C_2 to vary
- Let the particular solution $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$, now we solve for u_1 and u_2
  - To fully specify these functions we need 2 conditions; the first one will be the differential equation, and the second one will be chosen to make things convenient
- $y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$, choose $u_1' y_1 + u_2' y_2 = 0 \implies y_p' = u_1 y_1' + u_2 y_2' \implies y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$
- Subbing it back into the DE: $$\begin{aligned} & (u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2') + a(u_1 y_1' + u_2 y_2') + b(u_1 y_1 + u_2 y_2) = \phi(x) \\ \implies & (y_1'' + a y_1' + b y_1)u_1 + (y_2'' + a y_2' + b y_2)u_2 + u_1' y_1' + u_2' y_2' = \phi(x) \\ \implies & u_1' y_1' + u_2' y_2' = \phi(x) \end{aligned}$$
  - $\begin{cases} u_1' y_1' + u_2' y_2' = \phi(x) \\ u_1' y_1 + u_2' y_2 = 0 \end{cases}$ has 2 equations and 2 unknowns and allows us to solve for u_1 and u_2
  - To solve this in general: $u_1' = -\dfrac{y_2 \phi}{y_1 y_2' - y_2 y_1'}, u_2' = \dfrac{y_1 \phi}{y_1 y_2' - y_2 y_1'}$
- Example: $y'' + y' - 2y = e^x$
  - $r^2 + r - 2 = 0 \implies r_1 = -2, r_2 = 1 \implies y_c = C_1 e^x + C_2 e^{-2x} \implies y_1 = e^x, y_2 = e^{-2x}$
  - Impose $\begin{cases} u_1' e^x - 2u_2' e^{-2x} = e^x \\ u_1' e^x + u_2' e^{-2x} = 0 \end{cases}$
  - $u_2' e^{-2x} = -u_1' e^x \implies u_1' e^x + 2u_1' e^x = e^x \implies 3u_1' = 1 \implies u_1 = \dfrac{1}{3}x$
    - Note that we don't need to worry about the constant of integration since this would simply become a part of the constant in y_c
  - $u_2' e^{-2x} = -\dfrac{1}{3}e^x \implies u_2' = -\dfrac{1}{3}e^{3x} \implies u_2 = -\dfrac{1}{9}e^{3x}$
  - $y_p = \dfrac{1}{3}xe^x - \dfrac{1}{9}e^x \implies y = C_1 e^x + C_2 e^{-2x} + \dfrac{1}{3}xe^x - \dfrac{1}{9}e^x = C_1 e^x + C_2 e^{-2x} + \dfrac{1}{3}xe^x$
- Example: $y'' + y = 3\sin x \sin(2x)$
  - ϕ does not lead to an obvious trial solution, so using the method of undetermined coefficients is hard in this case
  - $r^2 + 1 = 0 \implies r = \pm i \implies y_c = A\cos x + B\sin x$
  - Let $y_p = u_1(x)\cos x + u_2(x)\sin x$
  - $\begin{cases} u_1' \cos x + u_2 \sin x = 0 \\ -u_1' \sin x + u_2' \cos x = 3\sin x \sin(2x) \end{cases}$
  - $$\begin{aligned} u_1' &= -\frac{y_2 \phi}{y_1 y_2' - y_2 y_1'} \\ &= -\frac{-3\sin^2 x \sin(2x)}{\cos^2 x + \sin^2 x} \\ &= -3\sin^2 x \sin(2x) \end{aligned}$$

- $u_1 = \int -3\sin^2 x \sin(2x)\,\mathrm{d}x$
  $= -3\int \sin^2 x(2\sin x\cos x)\,\mathrm{d}x$
  $= -6\int \sin^3 x\cos x\,\mathrm{d}x$
  $= -6\int v^3\,\mathrm{d}v$
  $= \frac{-3}{2}v^4$
  $= \frac{-3}{2}\sin^4 x$
- $u_2' = 3\cos x\sin x\sin(2x)$
- $u_2 = \int 3\cos x\sin x\sin(2x)\,\mathrm{d}x$
  $= 3\int \left(\frac{1}{2}\sin(2x)\right)\sin(2x)\,\mathrm{d}x$
  $= \frac{3}{2}\int \sin^2(2x)\,\mathrm{d}x$
  $= \frac{3}{2}\int \frac{1}{2}(1-\cos(4x))\,\mathrm{d}x$
  $= \frac{3}{4}\left(x - \frac{1}{4}\sin(4x)\right)$
- $y_p = -\frac{3}{2}\cos x\sin^4 x + \frac{3}{16}(4x - \sin(4x))\sin x \implies y = -\frac{3}{2}\cos x\sin^4 x + \frac{3}{16}(4x-\sin(4x))\sin x + A\cos x + B\sin x$