Lecture 33, Dec 3, 2021

Second Order Linear Homogeneous ODEs With Constant Coefficients: Cases

- Depending on the discriminant of the auxiliary equation, 3 cases can occur:
  1. $a^2 - 4b > 0$ – Two real and distinct roots, so the solutions are $e^{r_1 x}$ and $e^{r_2 x}$ and we have a general solution $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
  2. $a^2 - 4b = 0$ – Two equal roots, so one solution is $y_1 = e^{rx}$
     - The other solution turns out to be $y_2 = xe^{rx}$
     - Plugging in y_2 gets us $(xr^2 e^{rx} + re^{rx} + re^{rx}) + a(xre^{rx} + e^{rx}) + bxe^{rx} = (r^2 + ar + b)xe^{rx} + (2r + a)e^{rx} = 0$
  3. $a^2 - 4b < 0$ – Two complex roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$
     - $$\begin{aligned} y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \\ &= c_1 e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + c_2 e^{\alpha x}(\cos(\beta x) - i\sin(\beta x)) \\ &= e^{\alpha x}((c_1 + c_2)\cos(\beta x) + i(c_1 - c_2)\sin(\beta x)) \\ &= e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)) \end{aligned}$$
- To evaluate the constants we need known values
  - For an IVP this is the values of $y(0)$ and $y'(0)$, and for a boundary value problem this may be y at two values, y' at two values, or anything else
    * "You can't always fit a square peg into a round hole"
  - IVPs in this case always have unique solutions, while boundary value problems don't
- Example IVP: $y'' + y' - 2y = 0$, $y(0) = 2, y'(0) = 5$
  - Characteristic equation: $r^2 + r - 2 = 0 \implies r_1 = -2, r_2 = 1 \implies y = C_1 e^x + C_2 e^{-2x}$
    * $y' = C_1 e^x - 2C_2 e^{-2x}$
  - Plugging in initial values: $\begin{cases} 2 = C_1 + C_2 \\ 5 = C_1 - 2C_2 \end{cases} \implies \begin{cases} C_1 = 3 \\ C_2 = -1 \end{cases}$
  - $y = 3e^x - e^{-2x}$
- Example BVP: $y'' + 4y' + 5y = 0$, $y(0) = 1, y\left(\frac{\pi}{2}\right) = 0$
  - Characteristic equation: $r^2 + 4r + 5 = 0 \implies r = -2 \pm i \implies y = e^{-2x}(A\cos x + B\sin x)$
  - Apply boundary conditions: $\begin{cases} 1 = 1(A \cdot 1 + B \cdot 0) = A \implies A = 1 \\ 0 = e^{-\pi}(A \cdot 0 + B \cdot 1) = Be^{-\pi} \implies B = 0 \end{cases}$
  - $y = e^{-2x}\cos x$
  - If we change the boundary conditions to $y(0) = 1, y(\pi) = 0$, then the second equation becomes $0 = e^{-2\pi}(-A + B \cdot 0) = -Ae^{-2\pi} \implies A = 0$, but from the first equation $A = 1$, therefore these boundary conditions make the equation unsolvable

Nonhomogeneous Linear ODEs (Constant Coefficients)

- $y'' + ay' + by = \phi(x)$ is a nonhomogeneous linear ODE with constant coefficients
  - Its associated homogeneous linear ODE, the *complementary equation*, is $y'' + ay' + by = 0$
- Theorem: The general solution to a nonhomogeneous second order linear ODE with constant coefficients is $y(x) = y_p(x) + y_c(x)$ where $y_p(x)$ is a particular solution and $y_c(x)$ is the solution of the complementary equation
  - Proof: Given $y_{p_1}(x)$ and $y_{p_2}(x)$, we can show that the latter is a linear combination of the first and $y_c(x)$
    * Let $z = y_{p_1} - y_{p_2}$, if we can show that z is a solution to the complementary solution then we can prove this
    * Since $y_{p_2} = y_{p_1} - z \implies y'_{p_2} = y'_{p_1} - z'$ and $y''_{p_2} = y''_{p_1} - z''$

    * $$\begin{aligned} & y''_{p_2} + ay'_{p_2} + by_{p_2} = \phi(x) \\ \implies & (y''_{p_1} - z'') + a(y'_{p_1} - z') + b(y_{p_1} - z) = \phi(x) \\ \implies & (y''_{p_1} + ay'_{p_1} + by_{p_1}) - (z'' + az' + bz) = \phi(x) \\ \implies & z'' + az' + bz = (y''_{p_1} + ay'_{p_1} + by_{p_1}) - \phi(x) \\ \implies & z'' + az' + bz = 0 \end{aligned}$$
    * Therefore z is a solution to the complementary equation; thus any solution is a linear combination of one particular solution and the complementary solution
  - Thus we only need to find one particular solution to the nonhomogeneous equation, and then add on the solution to the complementary equation to obtain all solutions
- To find y_p, we can use one of the methods below:

Method of Undetermined Coefficients

- This method is easier to do but more limited
- Assume that y_p has the same form as $\phi(x)$, take derivatives and substitute, and then solve for the undetermined coefficients
- Example: $y'' - 6y' + 8y = x^2 + 2x$
  - Complementary equation is $y'' - 6y' + 8y = 0$, solving the auxiliary equation gets us $r_1 = 2, r_2 = 4$ so $y_c = C_1e^{2x} + C_2e^{4x}$
  - Here $\phi(x)$ is a second order polynomial so try y_p in the same form, so $y_p = Ax^2 + Bx + C$
    * $y'_p = 2Ax + B$, $y''_p = 2A$
  - Substitute the trial solution: $2A - 6(2Ax + B) + 8(Ax^2 + Bx + C) = x^2 + 2x \implies 8Ax^2 + (-12A + 8B)x + (2A - 6B + 8C) = x^2 + 2x$
    * From this we know $\begin{cases} 8A = 1 \\ -12A + 8B = 2 \\ 2A - 6B + 8C = 0 \end{cases}$
    * We can solve for $A = \dfrac{1}{8}, B = \dfrac{7}{16}, C = \dfrac{19}{64}$
  - $y_p = \dfrac{1}{8}x^2 + \dfrac{7}{16}x + \dfrac{19}{64}$
  - $y = \dfrac{1}{8}x^2 + \dfrac{7}{16}x + \dfrac{19}{64} + C_1e^{2x} + C_2e^{4x}$
- Examples of trial solutions:
  - $\phi(x) = e^{3x} \implies y_p = Ae^{3x}$
  - $\phi(x) = C\cos(kx)$ *or* $C\sin(kx)$ then $y_p = A\cos(kx) + B\sin(kx)$
  - $\phi(x) = x^2\sin(kx) \implies y_p = (Ax^2 + Bx + C)\cos(kx) + (Dx^2 + Ex + F)\sin(kx)$
- If $\phi(x)$ is a sum, then we can use the superposition principle: If $y'' + ay' + by = \phi_1(x) + \phi_2(x)$, then $\begin{cases} y''_{p_1} + ay'_{p_1} + by_{p_1} = \phi_1(x) \\ y''_{p_2} + ay'_{p_2} + by_{p_2} = \phi_2(x) \end{cases} \implies y_p = y_{p_1} + y_{p_2}$
- If the obvious trial solution is a multiple of y_c, then multiply y_p by x
  - Example: $y'' + y = \sin x$ has $y_c = C_1\cos x + C_2\sin x$, so try $y_p = Ax\cos x + Bx\sin x$
- Example: $y'' - y' - 6y = e^{-2x}$
  - $r^2 - r - 6 = 0 \implies r_1 = -2, r_2 = 3 \implies y_c = C_1e^{-2x} + C_2e^{3x}$
  - Try trial solution $y_p = Axe^{-2x} \implies y'_p = Ae^{-2x} - 2Axe^{-2x} \implies y''_p = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x} = (4Ax - 4A)e^{-2x}$
  - $(4Ax - 4A)e^{-2x} - (A - 2Ax) - 6Axe^{-2x} = e^{-2x} \implies \begin{cases} 4A + 2A - 6A = 0 \\ -4A - A = 1 \end{cases} \implies A = -\dfrac{1}{5}$
  - Therefore the general solution is $y = C_1e^{-2x} + C_2e^{3x} - \dfrac{1}{5}xe^{-2x}$
- This method guesses for the answer so it's not always possible to solve for the coefficients