Lecture 31, Nov 29, 2021

Linear Equations Continued

- Example: Logistic model $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$ with $M(t)$ and $k(t)$ as a function of time
  - First we have to make the equation linear since there is a P^2 term; substitute $z = \frac{1}{p} \implies P = \frac{1}{z} \implies \frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{z^2}\frac{\mathrm{d}z}{\mathrm{d}t}$
  - $-\frac{1}{z^2}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{k}{z}\left(1 - \frac{1}{ZM}\right) \implies \frac{\mathrm{d}z}{\mathrm{d}t} = -kz\left(1 - \frac{1}{ZM}\right) \implies \frac{\mathrm{d}z}{\mathrm{d}t} + kz = \frac{k}{M}$ and it is now linear
  - If k and M are constant, then integrating factor: $H(t) = kt \implies$ integrating factor is e^{kt}
  - $z = e^{-kt}\left[\int e^{kt}\frac{k}{M}\,\mathrm{d}t + C\right] = e^{-kt}\left[\frac{e^{kt}}{M} + C\right] = \frac{1}{M} + Ce^{-kt}$
  - Therefore $P = \frac{1}{z} = \frac{M}{1 + MCe^{-kt}}$
- Example: Solving an RC (resistor-capacitor) circuit
  - Voltage drop across a capacitor is $\frac{Q}{C}$ where Q is the charge in coulombs and C is capacitance in Farads
  - $E(t) = \frac{Q}{C} = IR$, but since $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, $E(t) = \frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t}$
  - In standard form, $Q' + \frac{1}{RC}Q = \frac{E(t)}{R}$
  - $H(t) = \int \frac{1}{RC}\,\mathrm{d}t = \frac{t}{RC}$ so the integrating factor is $e^{\frac{t}{RC}}$
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{t}{RC}}Q\right) = e^{\frac{t}{RC}}\frac{E(t)}{R}$
  - $Q = e^{-\frac{t}{RC}}\left(\int e^{\frac{t}{RC}}\frac{E(t)}{R}\,\mathrm{d}t + A\right)$
- A nonlinear equation in the form $y' + p(x)y = q(x)y^r$ can be made linear by substituting $u = y^{1-r}$
  - $u' = (1-r)y^{-r}$
  - $y' + p(x)y = q(x)y^r \implies (1-r)y^{-r}y' + (1-r)y^{-r}y = (1-r)y^{-r}q(x)y^r \implies u' + (1-r)p(x)u = (1-r)q(x)$
  - Now we can apply the integrating factor
  - Equations in this form are called *Bernoulli Equations*

Complex Numbers

- Our number system is missing solutions to equations such as $x^2 = -1$, and like we extended the reals with CORA, we need more axioms to extend our number system
- Calculus of complex number systems is called complex analysis
- Define the *imaginary unit* $i^2 = -1$ or $i = \sqrt{-1}$, and a *complex number* $z = a + ib$ where $a, b \in \mathbb{R}$ and $\mathrm{Re}(z) = a, \mathrm{Im}(z) = b$
- We can represent the complex number $a + ib$ as the point (a, b) on the complex plane (instead of a number line), where the horizontal axis is the real axis and the vertical axis is the imaginary axis
  - This is known as an *Argand Diagram*, and the plane is known as the complex plane, $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$
- Complex numbers can be represented in a polar form (like polar vectors)
  - The distance from the origin of a complex number is known as the *modulus*: $|z| = |a+ib| = \sqrt{a^2 + b^2}$
    - The modulus of a complex number is like the absolute value of a real number, and it is always real and nonnegative
  - The angle that the complex number makes with the real axis is the known as the *argument*: $\arg(z) = \theta$
    - Arguments are not unique; $\arg(z) = \theta \implies \arg(z) = \theta + 2k\pi$ where $k \in \mathbb{Z}$

  - $|z|\cos(\arg(z)) = a$, $|z|\sin(\arg(z)) = b$, and $\tan(\arg(z)) = \frac{b}{a}$ for $a \neq 0$
  - Let $r = |z|$ and $\theta = \arg(z)$, then the polar representation of z is $z = r\cos\theta + ir\sin\theta$
- Definition: The complex conjugate of $z = a + ib$ is $\bar{z} = a - ib$
  - In an Argand diagram the conjugate is a reflection across the real axis