Lecture 29, Nov 24, 2021

Solutions to Differential Equations

- Definition: A function is a solution of a DE if the substitution of the function and its derivatives lead to an identity.
  - Example: $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = 2x + \frac{4}{x}$ has solution $y(x) = 2x^2 + 7x - 4$
- The basic integration process is like a very simple differential equation; $\frac{\mathrm{d}y}{\mathrm{d}x} f(x) \implies y(x) = F(x) + C$
  - There is not one unique solution to a differential equation; generally the number of free parameters in the solution to a DE is equal to its order (e.g. first order DEs have 1 constant, second order DEs have 2)
  - The values of these constants can be determined by substituting in known values from initial conditions
- The *general solution* to an n-th order DE is an n-parameter family of solutions that include all the solutions to the DE.
- The *particular solution* is a member of the general solutions family for the DE and have specific values assigned to the constants from initial values
- An *initial value problem* consists of a DE and a number of initial values for the function; a *boundary value problem* is a DE and some known values not at 0

Separable Differential Equations

- Not all differential equations can be solved analytically, but some are easier to solve than others
- A general first-order DE can be expressed as $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y)$; if $F(x, y)$ can be separated into $F(x, y) = g(x)f(y)$ or $\frac{g(x)}{h(y)}$, then it is *separable* and can be solved easily
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)} \implies \int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$
    * $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)} \implies h(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \implies \int h(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x \implies \int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$
    * Note: Assume $\frac{\mathrm{d}}{\mathrm{d}y}H(y) = h(y)$ then $\frac{\mathrm{d}}{\mathrm{d}x}H(y) = h(y)\frac{\mathrm{d}y}{\mathrm{d}x}$ by the chain rule, which means that $\int h(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}H(y)\,\mathrm{d}y = H(y) = \int h(y)\,\mathrm{d}y$
- Example: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}e^x y^2$ with $y(0) = -1$
  - $\frac{1}{2}e^x y^2 = \frac{e^x}{\frac{2}{y^2}} \implies \int \frac{2}{y^2}\,\mathrm{d}y = \int e^x\,\mathrm{d}x \implies -2y^{-1} = e^x + C \implies y = -\frac{2}{e^x + C}$ is the general solution
  - Use the initial condition gets us $y = -\frac{2}{1 + C} = -1 \implies C = 1$ so the particular solution is $y = -\frac{2}{e^x + 1}$
  - Note this equation is nonlinear
- Example: Resistor-Inductor (RL) circuits
  - Suppose there is a resistor I and inductor L connected in series to a supply voltage of $E(t)$, then by Ohm's law the voltage drop across the resistor is $V = RI$ and the voltage drop across the inductor is $V = L\frac{\mathrm{d}I}{\mathrm{d}t}$, so $E(t) = L\frac{\mathrm{d}I}{\mathrm{d}t} + RI$ models the current I
  - Rewrite as $\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{V - RI}{L} \implies g(x) = 1, h(y) = \frac{L}{V - RI} \implies \int \frac{L}{V - RI}\,\mathrm{d}I = \int \mathrm{d}t$
  - After substituting in the numbers we can easily integrate and find a solution

Orthogonal Trajectories

- Given a family of curves, an *orthogonal trajectory* is a trajectory that passes through every one of these curves at 90 degrees
- The tangent of the trajectory is everywhere orthogonal to the tangent of the curves, so $f' = -\dfrac{1}{g'}$
- Example: $y^2 = kx^3$
  - Using implicit differentiation, $2yy' = 3kx^2 \implies y' = \dfrac{3kx^2}{2y}$
  - To get the general y', substitute back $y^2 = kx^3 \implies k = \dfrac{y^2}{x^3} \implies y' = \dfrac{3\left(\frac{y^2}{x^3}\right)x^2}{2y} = \dfrac{3}{2}\dfrac{y}{x}$
  - Then the tangent to the curve we want is $y' = \dfrac{-2x}{3y}$ and this is a separable differential equation so $\displaystyle\int 3y\,\mathrm{d}y = -\frac{2x}{}\mathrm{d}x \implies \frac{3y^2}{2} = -x^2 + C \implies 3y^2 + 2x^2 = 2C$
  - This is a family of ellipses