# Lecture 27, Nov 29, 2021

## Approximation of $e^x$

- Example: Show $e^x > 1 + x$ for $x > 0$
  - First show that $e^x > 1$ using integrals, and then take this back into the integral and show that $e^x > 1 + x$, then repeat this more to get $e^x > 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$
  - Key identity: $e^x = 1 + \int_0^x e^t \, \mathrm{d}t$
  - Note that $e^0 = 1$ and $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x > 0$ so it is always positive and increasing, and $e^x > 1$ for $x > 0$
  - $\int_0^x e^t \, \mathrm{d}t > \int_0^x \mathrm{d}t \implies 1 + \int_0^x e^t \, \mathrm{d}t > \int_0^x \mathrm{d}t + 1 \implies e^x > 1 + x$
  - We can continue this process making use of $e^x > 1 + x$, so $e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x (1 + t) \, \mathrm{d}t = 1 + x + \frac{x^2}{2}$
  - We can do this again one more time: $e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x (1 + t + \frac{t^2}{2}) \, \mathrm{d}t = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
  - Using induction we can show $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
    - Note we don't yet know that this infinite series converges to $e$

## General Exponential Function

- Definition: An irrational power $x^z = e^{z \ln x}$
  - Thus with this extended definition we have $x^{r+x} = x^r x^s$, $x^{r-s} = \frac{x^r}{x^s}$, $(x^r)^s = x^{rs}$ for $r, s \in \mathbb{R}$ provided $x > 0$
- We can extend the power rule to irrational powers
- Proof: $\frac{\mathrm{d}}{\mathrm{d}x}x^r = \frac{\mathrm{d}}{\mathrm{d}x}e^{r \ln x} = e^{r \ln x} \cdot \frac{r}{x} = \frac{x^r \cdot r}{x} = rx^{r-1}$, and with this we can also extend the reverse power rule for integrals
- Since $x^z$ is defined using the natural exponential, it has the same properties
- Example: $\frac{\mathrm{d}}{\mathrm{d}x}x^x$
  - $x^x = e^{x \ln x} \implies \frac{\mathrm{d}}{\mathrm{d}x}x^x = e^{x \ln x}\left(x \cdot \frac{1}{x} + \ln x\right) = x^x(1 + \ln x)$
- Exponentials with bases other than $p$: $\frac{\mathrm{d}}{\mathrm{d}x}p^x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}p^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x \ln p} = \ln(p)e^{x \ln p} = \ln(p)p^x$
- In general $\frac{\mathrm{d}}{\mathrm{d}x}p^u = p^u \ln(p)\frac{\mathrm{d}u}{\mathrm{d}x}$
- The integral form is $\int p^x \, \mathrm{d}x = \frac{1}{\ln p}p^x + C$ where $p > 0, p \neq 1$

## Logarithm With Other Bases

- Define $f(x) = \frac{\ln x}{\ln p}$ and $g(x) = p^x$, then $f(g(x)) = \frac{\ln p^x}{\ln p} = x\frac{\ln p}{\ln p} = x$ and $g(f(x)) = p^{\left(\frac{\ln x}{\ln p}\right)} = e^{\frac{\ln x}{\ln p} \cdot \ln p} = e^{\ln x} = x$, so they are inverses
- Define $\log_p(x) = \frac{\ln x}{\ln p}$ for $p > 0, p \neq 1$, note $\log_p(p^x) = x$ as they are inverses of each other
- $\frac{\mathrm{d}}{\mathrm{d}x}\log_p(u) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln u}{\ln p} = \frac{1}{\ln p} \cdot \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$

## Estimating $e$

- $\ln x = \int_1^x \frac{\mathrm{d}t}{t} \implies \ln\left(1 + \frac{1}{n}\right) = \int_1^{1+\frac{1}{n}} \frac{\mathrm{d}t}{t} < \int_1^{1+\frac{1}{n}} = 1 + \frac{1}{n} - 1 = \frac{1}{n}$
- $\ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \implies 1 + \frac{1}{n} < e^{\frac{1}{n}} \implies \left(1 + \frac{1}{n}\right)^n < e$
- To find an upper bound, $\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+\frac{1}{n}} \frac{1}{t}\,\mathrm{d}t > \int_1^{1+\frac{1}{n}} \frac{1}{1+\frac{1}{n}}\,\mathrm{d}t$, since $1 < t < \frac{1}{1+\frac{1}{n}}$
- Therefore $\ln\left(1 + \frac{1}{n}\right) > \left(\frac{1}{1+\frac{1}{n}}\right)\left(1 + \frac{1}{n} - 1\right) = \frac{1}{n+1} \implies \left(1 + \frac{1}{n}\right)^{n+1} > e$
- Combining these two we get $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$
- As $n$ tends to infinity, the difference between $n$ and $n+1$ becomes small and the two bounds close in on each other to converge to the true value of $e$
- $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ by a squeeze-theorem like argument