Lecture 26, Nov 17, 2021

Logarithmic Differentiation

- If we want to differentiate a lengthy product $g(x) = g_1(x)g_2(x)\cdots g_n(x)$, we can take the log of both sides to get $\ln|g(x)| = \ln|g_1(x)| + \ln|g_2(x)| + \cdots + \ln|g_n(x)|$, then differentiate $\frac{g'(x)}{g(x)} = \frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}$, and multiply by $g(x)$ to get $g'(x) = g(x)\sum_{i=1}^{n}\frac{g_i'(x)}{g_i(x)}$ (known as *logarithmic differentiation*)
- Example: $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^4(x-1)}{(x+2)(x^2+1)} = \frac{x^4(x-1)}{(x+2)(x^2+1)}\left[\frac{4x^3}{x^4} + \frac{1}{x-1} - \frac{1}{x+2} - \frac{2x}{x^2+1}\right]$
  - Notice that if a term is in the denominator it is subtracted instead, since the logarithm of that term is negative

The Natural Exponential

- Using the natural logarithm, we can extend the domain of the exponential to irrational numbers
- From the IVT we know that for some irrational z, $\ln x$ will take on that value at some point
- Definition: Let z be an irrational; then e^z is the unique number such that $\ln e^z = z$
- Definition: The exponential function $\exp x = e^x$, defined as the number such that $\ln e^x = x$
- Properties of the exponential:
  1. ln is the inverse of the exponential: $\ln e^x = x$ for $x \in \mathbb{R}$ as per our definition of the exponential and irrational powers
  2. $e^x > 0$, which comes from the fact that $\ln e^x$ is only defined for positive e^x
  3. $e^0 = 1$ from $\ln 1 = 0$
  4. $\lim_{x\to-\infty} e^x = 0$
  5. $e^{\ln x} = x$
  6. $e^{a+b} = e^a \cdot e^b$
     - From $\ln(e^a \cdot e^b) = \ln e^a + \ln e^b = a + b = \ln e^{a+b}$
  7. In a similar manner $e^{-b} = \frac{1}{e^b}$ and $e^{a-b} = \frac{e^a}{e^b}$, both of which come from the logarithm
  8. $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$
     - $\ln e^x = x \implies \frac{\mathrm{d}}{\mathrm{d}x}\ln e^x = \frac{1}{e^x}\frac{\mathrm{d}}{\mathrm{d}x}e^x = \frac{\mathrm{d}}{\mathrm{d}x}x = 1 \implies \frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$
  9. $\frac{\mathrm{d}}{\mathrm{d}x}e^{kx} = ke^{kx}$ from the chain rule
  10. $\int e^x\,\mathrm{d}x = e^x + C$
  11. $\int e^{g(x)}g'(x)\,\mathrm{d}x = e^{g(x)} + C$