Lecture 25, Nov 15, 2021

Properties of $\ln x$

- Further properties of $\ln x$:
  1. Defined on $(0, \infty)$, and $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x}$
  2. $\ln x$ is continuous since it is differentiable
  3. For all $x > 1$, $\ln x > 0$, since the integral area is always positive
  4. For $0 < x < 1$, $\ln x < 0$ as follows from the previous point
  5. $\ln(a + b) = \ln a + \ln b$
     - This can be proven using a different way
     - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x}$, and $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(ax) = \dfrac{1}{ax} \cdot a = \dfrac{1}{x}$
     - Since $\ln x$ and $\ln(ax)$ have same derivative, they differ by a constant, so
     - Therefore $\ln(ax) = \ln(x) + C$, and if we let $x = 1 \implies \ln a = 0 + C = C$, therefore $\ln(ax) = \ln(x) + \ln(a)$
  6. $\ln\left(x^{\frac{p}{q}}\right) = \dfrac{p}{q} \ln x$
     - We can show this in the same way as the one above
     - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln\left(x^{\frac{p}{q}}\right) = \dfrac{1}{x^{\frac{p}{q}}} \cdot \dfrac{p}{q} x^{\frac{p}{q}-1} = \dfrac{p}{q}\dfrac{1}{x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{p}{q} \ln x$
     - Therefore $\ln\left(x^{\frac{p}{q}}\right) = \dfrac{p}{q} \ln x + C$, and now if we let $x = 1$, then $0 = C$
  7. The range of $\ln x$ is $(-\infty, \infty)$ (i.e. it is unbounded)
     - Proof that ln is unbounded as $x \to \infty$: Show that for $M > 0$ imposed, there exists a x_0 such that $x > x_0 \implies \ln x > M$
       * Begin with $\ln 2 = \displaystyle\int_1^2 \frac{1}{t}\,\mathrm{d}t$, and since the integrand is always positive, $\ln 2 > 0$
       * Therefore, we can always find some positive n such that $n \ln 2 > M$ no matter how big M is, so choose $x_0 = 2^n$
       * When $x > x_0 = 2^n$, we have $\ln x > n \ln 2 > M$, therefore x is unbounded above
     - A similar argument follows for when $x \to 0$, therefore $x = 0$ is a vertical asymptote
  8. $\ln e = 1$
     - Since the $\ln x$ is unbounded and starts at 0 when $x = 1$, it must take on the value of 1 sometime, so we call this value e
  9. Convention $\ln x = \log_e(x)$
  10. $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x} > 0$ so it is increasing, $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \ln x = -\dfrac{1}{2x} < 0$ so it is concave down

Graphing Logarithms

- Chain rule with logarithms: $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln u = \dfrac{1}{u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$
- When graphing a logarithm note the argument can only be positive, and when it approaches zero, the value of the logarithm approaches negative infinity
- x intercepts when the argument is 1

Using Logarithms to Integrate and Differentiate

- $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$, absolute value because the domain of $\dfrac{1}{x}$ includes the negative numbers
- More generally, $\displaystyle\int \frac{g'(x)}{g(x)}\,\mathrm{d}x = \ln|g(x)| + C$ for $g(x) \neq 0$ (by substitution)

- Example: $\displaystyle \int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x$

$$
\begin{aligned}
&= -\int \frac{1}{u} \, \mathrm{d}u \\
&= -\ln|u| \\
&= -\ln|\cos x| + C \\
&= \ln|\sec x| + C
\end{aligned}
$$

- Example: $\displaystyle \int \cot x \, \mathrm{d}x = \int \frac{\cos x}{\sin x} \, \mathrm{d}x$

$$
\begin{aligned}
&= \int \frac{1}{u} \, \mathrm{d}u \\
&= \ln|u| + C \\
&= \ln|\sin x| + C
\end{aligned}
$$

- Example: $\displaystyle \int \sec x \, \mathrm{d}x = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x$

$$
\begin{aligned}
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \\
&= \int \frac{1}{u} \, \mathrm{d}u \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C
\end{aligned}
$$