Lecture 23, Nov 3, 2021

Volumes by Cylindrical Shells

- For volumes that may be difficult to integrate with the disk and washer method
- E.g. $y = f(a)$ rotated around the y axis
- Each slice of the area under the curve of f is going to be rotated and forms a hollow cylindrical shell
  - Height: $f(x_i^*)$, radius x_i^*, and thickness Δx; the volume of each shell is the area of the rectangle multiplied by the circumference at that location, $\Delta V = f(x_i^*)\Delta x_i \cdot 2\pi x_i^*$
  - Taking the limit as the number of shells approaches infinity, $V = \displaystyle\int_a^b 2\pi x f(x)\,\mathrm{d}x$
- If the region is defined by the area between two curves then $V = \displaystyle\int_a^b 2\pi x(f(x) - g(x))\,\mathrm{d}x$
- Similarly we can also rotate around the x axis instead by replacing all the x with y
  - Note that in the washer method, rotation around the x axis leads to integration wrt x, but for the shell method it's the opposite; rotation around the x axis leads to integration wrt y
- Example: Rotate the area between $y^2 - x^2 = 1$ and $y = 2$ about the x axis
  - We can take advantage of symmetry
  - For a given y value, $x = \sqrt{y^2 - 1}$; the minimum y occurs when $x = 0 \implies y = 1$
  - This leads to the integral $V = 2\displaystyle\int_1^2 2\pi y(\sqrt{y^2 - 1})\,\mathrm{d}y$
  - To evaluate, substitute $u = y^2 - 1 \implies \mathrm{d}u = 2y\,\mathrm{d}y \implies$
$$\begin{aligned} V &= 4\pi\int_0^3 \sqrt{u}\,\mathrm{d}u \\ &= 2\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3 \\ &= 4\sqrt{3}\pi \end{aligned}$$
- If rotating about an axis that is parallel to one of the principle axes, subtract an offset $V = \displaystyle\int_a^b 2\pi x(f(x) - k)\,\mathrm{d}x$

Average Value of a Function

- Since the average of a set of discrete numbers is $a_{avg} = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$, the average of a function taken at discrete points is $f_{avg} = \dfrac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$
- If we consider a uniform partition, then $\Delta x = \dfrac{b-a}{n} \implies n = \dfrac{b-a}{\Delta x}$
- If we substitute this into the equation above then $f_{avg} = \dfrac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\Delta x}{b-a}$
- Take the limit $n \to \infty$, the sum turns into an integral $f_{avg} \equiv \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$ (note this is a definition of the average)
- This is defined as the average value of f
- This leads to the *Mean Value Theorem for Integrals*: For f continuous on $[a, b]$, there exists $c \in [a, b]$ such that $f(c) = f_{avg} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$; a continuous function must equal its average value at some point
  - Proof: Let $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$, then by the MVT there exists c such that $F'(c) = \dfrac{F(b) - F(a)}{b-a}$ for $c \in [a, b]$; therefore $F'(c) \implies f(c) = \dfrac{\int_a^b f(t)\,\mathrm{d}t - \int_a^a f(t)\,\mathrm{d}t}{b-a} = \dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,\mathrm{d}t$
  - This is also a special case of the second MVT for $g(x) = 1$

- The *Second Mean Value Theorem* for integrals: For nonnegative g, $\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x$ where $c \in [a, b]$
  - Proof: By the EVT f has a maximum and minimum $m \le f(x) \le M$; since g is nonnegative, $mg(x) \le f(x)g(x) \le Mg(x) \implies m\int_a^b g(x)\,\mathrm{d}x \le \int_a^b f(x)g(x)\,\mathrm{d}x \le M\int_a^b g(x)\,\mathrm{d}x$
    * If g is everywhere zero then the theorem holds true since $0 = 0$
    * Otherwise we can divide by $\int_a^b g(x)\,\mathrm{d}x$ to get $m \le \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \le M$
    * By the IVT, there exists a $c \in [a, b]$ such that $f(c) = \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}$, since this value is between m and M and f takes on all values in that interval
    * Therefore there exists $c \in [a, b]$ such that $f(c)\int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)g(x)\,\mathrm{d}x$
  - Notes: $f(c)$ is *not* the average of f as from the first MVT; this is a weighted average where $g(x)$ is the weight
  - Think of it like a centre of mass where $g(x)$ describes the density, so $\frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}$ is the weighted average/centre of mass