Lecture 21, Oct 29, 2021

Area Between Curves

- Suppose $f(x) \geq g(x)$ and continuous on $x \in [a, b]$, then the area between them can be partitioned just like the area under a single curve into $A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x_i$; taking the limit, $A = \lim_{\|P\| \to 0} \sum_{i=0}^{n} [f(x_i^*) - g(x_i^*)] \Delta x_i = \int_a^b [f(x) - g(x)] \, \mathrm{d}x$
- For positive functions this can be interpreted as the difference between the areas under two curves
- Example: Find the area between $y = x + 2$ and $y = 4 - x^2$
  - Find the intersection: $4 - x^2 = x + 2 \implies x^2 + x - 2 = 0 \implies (x-1)(x+2) = 0$, so the intersections are at $(1, 3)$ and $(-2, 0)$
  - On this interval $x + 2 > 4 - x^2$
  - $$\begin{aligned} \int_{-2}^{1} (4 - x^2 - (x+2)) \, \mathrm{d}x &= \int_{-2}^{1} (-x^2 - x + 2) \, \mathrm{d}x \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1} \\ &= \frac{9}{2} \end{aligned}$$
- Need to be careful when curves cross since the curve that's the upper bound may change; generally the area is $A = \int_a^b |f(x) - g(x)| \, \mathrm{d}x$
- The independent variable does not have to be x

Volumes

- To find volumes, we can break up the region into very small slices and add up all the little bits of area
- $V_i \approx A_i \Delta x_i \implies V \approx \sum_{i=1}^{n} A(x_i^*) \Delta x_i$, and in the limit $V = \int_a^b A(x) \, \mathrm{d}x$
- Example: Rectangular pyramid on the x axis, h units tall, with a base width of r
  - The radius at each x is $\frac{r}{2h} x$, since if you look at it from the side, the edge of the pyramid is $\frac{r}{2}$ units above the x axis and h units from the y axis, so the slope is $\frac{r}{2h}$
  - Therefore $A(x) = \left(\frac{rx}{h} \right)^2 = \frac{r^2 x^2}{h^2}$
  - $$\begin{aligned} V &= \int_0^h \frac{r^2 x^2}{h^2} \, \mathrm{d}x \\ &= \frac{r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h \\ &= \frac{1}{3} r^2 h \end{aligned}$$
- Example: Calculate the volume of a sphere of radius r, at the origin $(x^2 + y^2 = r^2)$
  - If we take a slice, the radius here would be $y = \sqrt{r^2 - x^2}$, so the area of the slice would be $A = \pi(r^2 - x^2)$
  - $$\begin{aligned} V &= \int_{-r}^{r} \pi(r^2 - x^2) \, \mathrm{d}x \\ &= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \end{aligned}$$