Lecture 20, Oct 27, 2021

The Substitution Rule

- We can use the chain rule in reverse, known as u-substitution
- Since $\frac{\mathrm{d}}{\mathrm{d}x}f(g(x))) = f'(g(x))g'(x)$, we can reverse this
- If we have $\int f(g(x))g'(x)\,\mathrm{d}x$, make the substitution that $\mathrm{d}u = g'(x)\,\mathrm{d}x$, then $$\begin{aligned}\int f(g(x))g'(x)\,\mathrm{d}x &= \int f(u)\,\mathrm{d}u \\ &= F(u) + C \\ &= F(g(x)) + C\end{aligned}$$
- Example: $\int \frac{x}{\sqrt{x^2-4}}\,\mathrm{d}x$, let $u = x^2 - u$ and $\mathrm{d}u = 2x\,\mathrm{d}x$, then $$\begin{aligned}\int \frac{x}{\sqrt{x^2-4}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= \frac{1}{2}\cdot 2u^{\frac{1}{2}} + C \\ &= \sqrt{x^2-4} + C\end{aligned}$$
- Recognizing the appropriate substitution to make requires experience
- For definite integrals, we can eliminate the back substitution step by using a *change of variables formula*: $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$
  - Proof: Given $F' = f$, $$\begin{aligned}&\int_a^b f(g(x))g'(x)\,\mathrm{d}x \\ &= \int_a^b F'(g(x))g'(x)\,\mathrm{d}x \\ &= [F(g(x))]_a^b \\ &= F(g(b)) - F(g(a)) \\ &= [F(u)]_{g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u\end{aligned}$$
  - Example: $\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x$, let $u = x^2+1, \mathrm{d}u = 2x\,\mathrm{d}x \implies u(0) = 1, u(1) = 2$, so $$\begin{aligned}\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x &= \int_1^2 \frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u \\ &= \left[\frac{u^{\frac{3}{2}}}{3}\right]_1^2 \\ &= \frac{2\sqrt{2}}{3} - \frac{1}{3}\end{aligned}$$
- Substitution can be used even in instances that aren't exactly $f(g(x))g'(x)$ to simplify things
  - Example: $\int x^2(2x+1)^2\,\mathrm{d}x$, we can substitute $u = 2x+1, \mathrm{d}u = 2\,\mathrm{d}x$, and $x = \frac{1}{2}(u-1)$, so the integral becomes $\frac{1}{2}\int \left(\frac{1}{2}(u-1)\right)^2 u^4\,\mathrm{d}u$. This still needs to be expanded out, but now we only have to expand the quadratic instead of the quartic

Using Symmetry to Evaluate Integrals

1. If f is odd on $[-a, a]$ then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$

2. If f is even on $[-a, a]$ then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{2} f(x)\,\mathrm{d}x$