Lecture 2 (Online)

Defining Numbers

- Numbers are the most basic elements and math and thus can't be defined explicitly
- Numbers are defined implicitly by imposing the **axioms**, basic rules, that they have to satisfy
- Some axioms:
  1. Commutative law $x + y = y + x$
  2. Associative law $(x + y) + z = x + (y + z)$
  3. Distributive law $x(y + z) = xy + xz$
  4. Existence of additive identity 0 and multiplicative identity 1
  5. Existence of additive inverses $-x$ and multiplicative inverses x^{-1} for $x \neq 0$
- There should be as few axioms as possible as axioms are unprovable; the more axioms, the more the risk of contradiction
- Theorems arise from definitions and axioms and are provable: Example: $4 = 2 + 2$; there is no limit to the number of theorems
- Some definitions:
  1. Positive integers ("natural numbers"): $1, 2, 3, \ldots$; $2 \equiv 1 + 1$
  2. Rational numbers: $\frac{a}{b} \equiv a \cdot \frac{1}{b}$ where a and b are integers and $b \neq 0$
     - The axiom for the existence of inverses combined with this creates rational numbers
  3. Roots, other algebraic (root of a polynomial) irrational numbers; e.g. $q^2 - 2 = 0$
     - Axiom 5 only creates rational numbers, so a new one is needed for irrational numbers
  4. Transcendental irrationals; e.g. π or e
- Axiom: *Completeness of the Reals Axiom* (CORA): Every non-empty set of real numbers that is bounded above has a least upper bound among the real numbers; this creates irrationals
  - A set of real numbers S_1 is bounded above iff there exists some ubS_1 such that $x \leq \mathrm{ubS}_1$ for all $x \in S_1$; the least upper bound lubS_1 is the least of the ubS_1
    * The least upper bound does not have to be in S
  - Example: $S_2 = \{x : x^2 < 2\}$
    * lubS_2 intuitively would satisfy $\mathrm{lubS}_2^2 = 2$ and so $\mathrm{lubS}_2 = \sqrt{2}$
    * By imposing CORA, $\sqrt{2}$ and all irrationals (including transcendental irrationals) have been created
    * However proving $\mathrm{lubS}_2 = \sqrt{2}$ rigorously is very difficult