Lecture 19, Oct 25, 2021

The Fundamental Theorem of Calculus

- Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$
  - Example: $f(x) = x \implies \int_0^x t\,\mathrm{d}t = \frac{1}{2}x^2 = F(x)$
  - Notice that $F'(x) = x = f(x)$; is this true in general?
- F is the area under $f(x)$ from a to x, so for small h, $F(x+h) - F(x)$ is approximately the area of the small rectangle of width h and height $f(x)$, so intuitively $\frac{F(x+h)-F(x)}{h} \approx f(x)$
- Theorem: Let f be a continuous function on $[a, b]$, defined $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then F is continuous on $[a, b]$, differentiable on (a, b) and has derivative $F'(x) = f(x)$ for all $x \in (a, b)$
  - Proof: For $x, x+h \in (a, b)$, $F(x+h) - F(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t = \int_x^{x+h} f(t)\,\mathrm{d}t$
  - Then $\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t$
  - Consider $h > 0$:
    * By the EVT, there exists minimum $f(u) = m$ and maximum $f(v) = M$ for $u, v \in [x, x+h]$; then by the order theorems,
    $$\begin{aligned} f(u)h &\leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq f(v)h \\ \implies f(u) &\leq \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \leq f(v) \\ \implies f(u) &\leq \frac{F(x+h)-F(x)}{h} \leq f(v) \end{aligned}$$
    * As $x \leq u, v \leq x+h$, $\lim_{h\to 0} u = \lim_{h\to 0} v = x \implies \lim_{h\to 0} f(u) = \lim_{h\to 0} f(v) = f(x)$
    * Thus by the squeeze theorem $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$, i.e. $F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$
- The fundamental theorem of calculus: Let f be continuous on $[a, b]$, then if G is any antiderivative of f on $[a, b]$ then $\int_a^b f(t)\,\mathrm{d}t = G(b) - G(a)$
  - Proof: $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f by the previous theorem; then $F'(x) = G'(x) \implies F(x) = G(x) + C$
  - $F(a) = \int_a^a f(t)\,\mathrm{d}t = 0 \implies G(a) + C = 0 \implies C = -G(a) \implies F(x) = G(x) - G(a)$
  - Then $\int_a^b f(t)\,\mathrm{d}t = F(b) = G(b) - G(a)$

Net Change and Indefinite Integrals

- $\int_a^b F'(x)\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x \approx \sum \mathrm{d}F = \Delta F = F(b) - F(a)$
- By taking an integral, we are summing up very small pieces of change to get the net change
- Example: Integrating the velocity $v(t)$ to get a change in position
- We can leave out the bounds and get $\int f(x)\,\mathrm{d}x = F(x) + C$, which is an *indefinite integral*
- An indefinite integral is a family of curves since the value of C can vary

- Example: $\int (2+\tan^2\theta)\,\mathrm{d}\theta = \int (1+1+\tan^2\theta)\,\mathrm{d}\theta$
$$= \int (1+\sec^2\theta)\,\mathrm{d}\theta$$
$$= \theta + \tan\theta + C$$
- If we are given an initial condition, we can use that to determine a value for C
- Example: Suppose that a particle has an acceleration of $a(t) = 3 - t$, and $x(0) = 2, x(3) = -1$, where is the particle at $t = 6$? How far has the particle travelled in the 6 seconds?
  - $v(t) = \int a(t)\,\mathrm{d}t = 3t - \frac{1}{2}t^2 + C$
  - $x(t) = \int v(t)\,\mathrm{d}t = \frac{3}{2}t^2 - \frac{1}{6}t^3 + Ct + k$
  - From the known values of $x(t)$, we can find C and k
  - $x(0) = k = 2 \implies k = 2$
  - $x(3) = \frac{3}{2}9 - \frac{1}{6}27 + 3C + 2 = -1 \implies C = -4$
  - Therefore $x(t) = \frac{3}{2}t^2 - \frac{1}{6}t^3 - 4t + 2$, thus at $t = 6$, the particle is at $x(t) = -4$
  - To find the distance travelled, we need to integrate the absolute velocity since the particle can have a negative velocity between $t = 0$ to 6
  - $s = \int_0^6 |v(t)|\,\mathrm{d}t = \int_0^6 \left|-\frac{1}{2}(t-2)(t-4)\right|\,\mathrm{d}t$
  - The particle changes direction twice; for $t \in (0,2)$, $v(t) < 0$; for $t \in (2,4)$, $v(t) > 0$; for $t > 4$, $v(t) < 0$, so we can use this to break up the integral
  - $\int_0^6 |v(t)|\,\mathrm{d}t = \int_0^2 -v(t)\,\mathrm{d}t + \int_2^4 v(t)\,\mathrm{d}t + \int_4^6 -v(t)\,\mathrm{d}t$
$$= [-x(t)]_0^2 + [x(t)]_2^4 - [x(t)]_4^6$$
$$= x(0) - x(2) + x(4) - x(2) + x(4) - x(6)$$
$$= 7 + \frac{1}{3}$$