Lecture 18, Oct 22, 2021

Defining the Definite Integral

- Definition: If f is a function defined on $[a, b]$, let P be a partition of $[a, b]$ with partition points $a = x_0 < x_1 < \cdots < x_n = b$, choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$, and $\|P\| = \max\{\Delta x_i\}$, then the *definite integral* of f from a to b is $\displaystyle\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ if the limit exists
  - This is called the *Riemann* definition of an integral, since $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is a *Riemann sum*
  - More precisely $\displaystyle\int_a^b f(x)\,\mathrm{d}x = I \iff \forall\varepsilon > 0, \exists\delta > 0$ such that $\|P\| < \delta \implies \left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \varepsilon$ for all partitions P of $[a, b]$ and all possible choices of $x_i^* \in [x_{i-1}, x_i]$
  - An integral doesn't need to represent an area; it can also represent other things and is a very general process
  - If the integral does represent an area, then it is either the area under a curve if the function is always positive, or a difference of areas if the function becomes negative at some point
- If $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ exists, then f is *integrable* over the interval $[a, b]$
- Even though the limit might not be evaluable analytically, the definite integral can still be approximated to any degree of accuracy desired
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$ and $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
- Theorem: Piecewise continuity implies integrability
  - Piecewise continuity means that there is only a finite number of jump discontinuities (this does not include infinite discontinuities)
  - Proof requires more background in series and sequences so comes later
  - If f goes to infinity at some point in the interval then the integral may or may not exist
- Practically, for continuous functions, we assume:
  1. Regular partition $\Delta x = \Delta x_i = \dfrac{b-a}{n}$
  2. Right hand or left hand end point: $x_i^* = x_i = a + i\Delta x = a + i\dfrac{b-a}{n}$
- With these assumptions, $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}$

Integral Properties

- Just like limits, we have theorems for integrals; in fact, because the definite integral is defined as a limit, many of these are just limit properties
- Properties of integrals:
  1. Constant: $\displaystyle\int_a^b c\,\mathrm{d}x = c(b-a)$
  2. Sum: $\displaystyle\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
  3. Constant multiple: $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$
  4. Combining bounds: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^z f(x)\,\mathrm{d}x + \int_z^b f(x)\,\mathrm{d}x$, note that z does not have to be between a and b!
- Order properties:
  1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$
4. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$