Lecture 17, Oct 20, 2021

Summation Theorems

1. Constant multiple: $\sum \alpha a_i = \alpha \sum a_i$
2. Additivity/distributive: $\sum (a_i + b_i) = \sum a_i + \sum b_i$
3. $\sum_{i=1}^{n} 1 = n$
4. $\sum_{i=1}^{n} \alpha = \alpha n$
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
8. $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Limits of Sums

- If $f(n) = \sum_{i=1}^{n} a_i$, then we can take $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$
- Example:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{5}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{5}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{5}{6} \lim_{n\to\infty} \frac{2n^3+3n^2+n}{n^3} \\
&= \frac{5}{6} \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&= \frac{5}{3}
\end{aligned}$$

Area Under a Curve

- How do we define the area under a curve?
- Definition: A *partition* is a way to divide a closed interval $[a, b]$ into a finite number of subsets, including a and b
  - e.g. $[x_0, x_1], [x_1, x_2], \cdots$
  - We can then define $\Delta x_i \equiv x_i - x_{i-1}$
- Definition: The *norm* of a partition P is $\|P\|$, defined as the length of the longest subinterval
- In every subinterval there is an x_i^* which lies within the interval, which makes the area of each rectangle $A_i = f(x_i^*)\Delta x_i$
- The total area is $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$
  - We can't simply make $n \to \infty$ because the partitions may not be equal in size

- In practice we usually work with equal sized intervals, but sometimes in numerical integration it might be useful to vary the size of the subintervals
- Example: $y = \cos x$, $0 \leq x \leq b \leq \frac{\pi}{2}$ with a regular partition $\Delta x_i = \frac{b}{n} = \|P\|$