Lecture 16, Oct 18, 2021

Curve Sketching

- Checklist:
  1. Find:
     - Domain, range
     - $\lim_{x \to \pm\infty} f(x)$
     - End points if they exist
     - Vertical/horizontal/slant asymptotes
  2. Find intercepts (x value where $f = 0$, f value where $x = 0$)
  3. Establish whether $f(x)$ is:
     - Even or odd
     - Periodic
  4. Find $f'(x)$, then:
     - Find all critical points c and $f(c)$ and find local max/min
     - Find ranges where $f(x)$ is increasing and decreasing
     - Find vertical tangents (reminder: vertical tangent is when derivative is infinity but function is continuous)
  5. Find $f''(x)$, then:
     - Find where $f(x)$ is concave up/down and points of inflection
       * Note: $f(x)$ has to be continuous at a point of inflection
     - Use the second derivative test to confirm local max/min from the last step
  6. Find the absolute maximum/minimum if they exist
- Example: $f(x) = 2x^3 - 3x^2$ (note $f(x) = x^2(2x-3)$)
  1. Domain is all x, no asymptotes, no end points, $\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = -\infty$ so range is all reals
  2. $f(x) = 0$ when $x = 0, \frac{3}{2}$, $x = 0$ when $f = 0$
  3. Not symmetric or periodic
  4. $f'(x) = 6x^2 - 6x = 6x(x-1)$
     - Critical points: $x = 0, 1$
     - $f' > 0$ for $x > 1 \implies f(x)$ increasing for $x > 1$
     - $f' < 0$ for $0 < x < 1 \implies f(x)$ decreasing for $0 < x < 1$
     - $f' > 0$ for $x < 0 \implies f(x)$ increasing for $x < 0$
     - $f(1)$ is a local minimum, $f(0)$ is a local max
     - No vertical tangents or cusps
  5. $f''(x) = 12x - 6 = 6(2x-1)$
     - Inflection point at $x = \frac{1}{2}$
     - $f'' > 0$ for $x > \frac{1}{2}$ so f is concave up for $x > \frac{1}{2}$
     - $f'' < 0$ for $x < \frac{1}{2}$ so f is concave down for $x < \frac{1}{2}$
  6. No absolute max/min since range is all reals

Numerical Root Finding Methods

- Sometimes we might want to find the roots of a complicated polynomial; for polynomials of degree 5 and above no formula exists, but there exist numerical methods that can get us approximate values
- Method of Successive Bisections
  - By trial and error, find $a < b$ such that $f(a) > 0 > f(b)$ or $f(a) < 0 < f(b)$
  - If f is continuous, then by the IVT, $f(c) = 0$ for some $a < c < b$
  - For each step:
    1. Calculate the halfway point between $a_i + b_i$, denoted x_{hi}

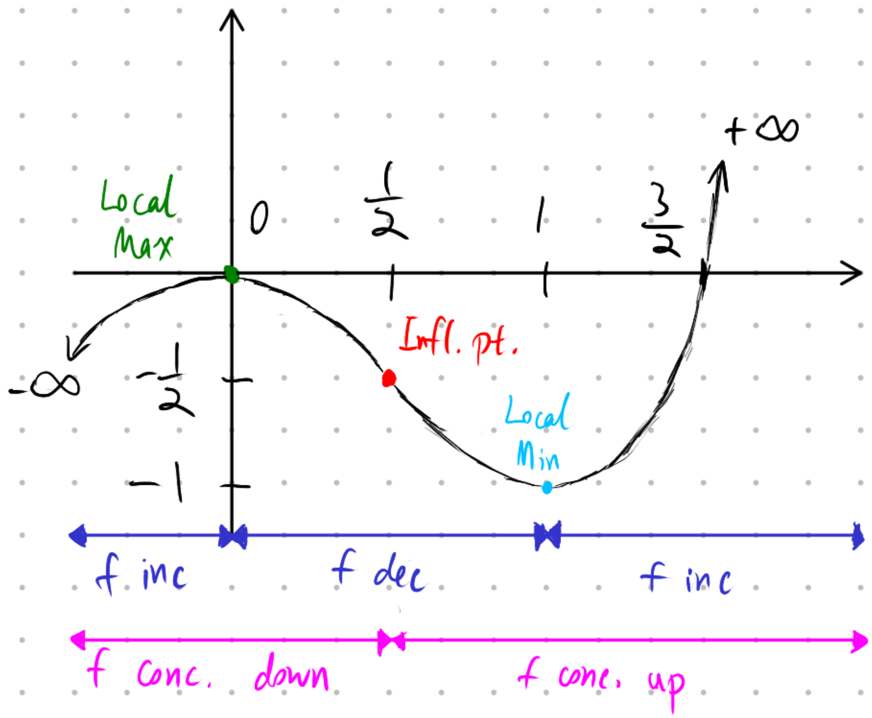

Figure 1: Graph of $f(x) = 2x^3 - 3x^2$

2. Calculate $f(x_{hi})$; if $f(x_{hi}) > 0$ then we can pick $a_{i+1} = x_{hi}$ and keep b the same
   * Conversely if $f(x_{hi}) < 0$ we keep a the same but pick $b_{i+1} = x_{hi}$

- For each step the possible range for c is cut in half; eventually the range is small enough that the difference between a and b does not matter
- This method always converges

- Newton's Method
  - Easier to use and faster
  - However, $f(x)$ must be differentiable, and this method may not converge if initial guess is too far off
  - Procedure:
    1. Start with an initial guess of x_1
       * This must be a good guess, otherwise will not converge
    2. Compute $f(x_1)$ and $f'(x_1)$ to find the tangent line: $y_t(x) = f(x_1) + f'(x_1)(x - x_1)$
    3. Find where the tangent line approximation intersects the x-axis: $f(x_1) + f'(x_1)(x_2 - x_1) = 0 \implies x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
    4. Repeat the procedure until estimate is as close to the true root as desired
  - If the tangent line at x is not a good approximation of the function's behaviour, then the next iteration can give a worse number (e.g. if there is a local max between your guess and the root)
  - This method may also fail for some particular functions, e.g. $x^{\frac{1}{3}}$
- Generally, try Newton's Method first, and if it does not converge, try different guesses or try the bisection method

Antiderivatives

- Definition: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$
- Unlike derivatives, not every function has an antiderivative; (educated) guessing is always involved
- The test for whether $F(x)$ is an antiderivative of $f(x)$ is the rigorous part, but finding $F(x)$ itself is not

(like limits)

- $F(x) + C$ is the most general antiderivative of $f(x)$ because antiderivatives are not unique
- Properties to be memorized:
  1. $af(x) \to aF(x) + C$
  2. $f(x) + g(x) \to F(x) + G(x) + C$
  3. $x^n \to \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$
  4. $\cos x \to \sin x + C$
  5. $\sin x \to -\cos x + C$
  6. $\sec^2 x \to \tan x + C$