Lecture 15, Oct 15, 2021

Horizontal Asymptotes/Limits at Infinity

- Definition: $\lim_{x\to\infty} f(x) = L$ means that $\forall\varepsilon > 0, \exists A$ such that $x > A \implies |f(x) - L| < \varepsilon$; this is a *horizontal asymptote* of $f(x)$
  - i.e. we can find f values as close to L as possible by going to large enough x values
- Limit theorems also exists for limits at infinity just like regular limits
- Theorem: $\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$
  - Proof: For all $\varepsilon > 0$, choose $A = \varepsilon^{-\frac{1}{r}}$
    * $$\begin{aligned} & x > A = \varepsilon^{-\frac{1}{r}} \\ \implies & x^r > \varepsilon^{-1} \\ \implies & \frac{1}{x^r} < \varepsilon \\ \implies & \left|\frac{1}{x^r}\right| < \varepsilon \end{aligned}$$
    * Thus $x > A \implies \left|\frac{1}{x^r} - 0\right| < \varepsilon$, so $\lim_{x\to\infty} \frac{1}{x^r} = 0$

Slant Asymptotes

- Definition: If $\lim_{x\to\infty} [f(x) - (mx + b)] = 0$, then the line $y = mx = b$ is a slant asymptote to $f(x)$ at $+\infty$
  - As always this does not mean that the limit at infinity exists, but it gives us more information than simply saying that $\lim_{x\to\infty} f(x)$ DNE

Max/Min Problems

- General checklist for max/min problems:
  1. Critical points
     - Find derivative, second derivative, determine concavity if it helps
  2. Endpoints
     - Find the domain, as constrained by the question statement
     - Find value at end points
  3. Check interior points
     - Find value at internal local extrema
  4. Check limit at $\pm\infty$ if the question allows it
  5. Compare values and decide the max/min
     - Make sure to state the answer!
- Example: A string 28cm long is to be cut into 2 parts to make a square and a circle. Where should the cut be made to get a) the maximum area and b) the minimum area?
  - Let the square have side x and the circle have radius y
  - We know that $28 = 4x + 2\pi y$ therefore we can relate x and y by $y = 7 - \frac{\pi}{2}x$, which allows us to eliminate one of the variables
  - The total area is then $A(x, y) = x^2 + \pi y^2 \implies A(y) = \left(7 - \frac{\pi}{2}y\right)^2 + \pi y^2$
  - At this point we need to **establish the domain** since the max/min can be at an endpoint
    * $y \geq 0$ and $x \geq 0$ due to physical constraints, so the domain is $\left[0, \frac{14}{\pi}\right]$
  - $A(0) = 49$, $A\left(\frac{14}{\pi}\right) = \pi\frac{14^2}{\pi^2} \approx 60$
  - Note A is a polynomial of y and therefore is continuous by the polynomial continuity theorem; therefore by the EVT for this *closed* interval there is an absolute max/min

    * Sometimes there is not an absolute max/min!
    * Note the importance of establishing the domain right away, so we know the function is continuous over the closed interval
- $A'(y) = -\pi\left(7 - \frac{pi}{2}y\right) + 2\pi y = -7\pi + y\left(\frac{\pi^2}{2} + 2\pi\right)$
- $A''(y) = \frac{\pi^2}{2} + 2\pi > 0$ therefore $A(y)$ is concave up
- $A'(y) = 0 \implies y_{crit} = \frac{7}{2 + \frac{\pi}{2}} \approx 2$
- $x_{crit} \approx 7 - \pi \approx 4$
- $A(y_{crit}) \approx 28$
- Since $A\left(\frac{14}{\pi}\right) > A(0) > A(y_{crit})$, the absolute max is $A\left(\frac{14}{\pi}\right)$ and the absolute min is $A\left(\frac{14}{\pi}\right)$
    * i.e. the max occurs when not cutting the string and using it all for the circle; the min occurs when cutting the string at $\frac{14}{\pi}$ (about 12 and 16cm lengths) for the square and circle respectively
- Since A' is a polynomial it always exists