Lecture 14, Oct 13, 2021

Bounding Estimations Using the Mean Value Theorem

- From last lecture we used differentials to estimate values: $29^{\frac{1}{3}} \approx 3.074$ using $f(x) = \sqrt[3]{x}$ and $x = 27$, $\Delta x = 2$
- Now we can use the MVT to bracket our estimate
- If we apply MVT to $f(x) = \sqrt[3]{x}$ on $[27, 29]$:
  - There is some $c \in (27, 29)$ such that $f'(c) = \dfrac{\sqrt[3]{29} - 3}{2}$
  - Since $f'(c) = \dfrac{1}{3}c^{-\frac{2}{3}}$
  - Therefore $\sqrt[3]{29} = \dfrac{2}{3}c^{-\frac{2}{3}} + 3$
  - Since we know $27 < c < 29$, we now have bounds on $\sqrt[3]{29}$; when $c = 27$, $\dfrac{2}{3}c^{-\frac{2}{3}} + 3 = \dfrac{2}{3} \cdot \dfrac{1}{9} + 3 = \dfrac{2}{27} + 3$ so c cannot exceed that
  - When $c = 29$ we can estimate $29^{-\frac{2}{3}}$, since $29 < 64 \implies 29^{\frac{2}{3}} < 64^{\frac{2}{3}} = 16 \implies 29^{-\frac{2}{3}} > \dfrac{1}{16}$ so $\sqrt[3]{29} < \dfrac{2}{3} \cdot \dfrac{1}{16} + 3$
  - Therefore $3.0416 < \sqrt[3]{29} < 3.074$

Derivatives and Graphing: 4 Quick Tests

1. Increasing/decreasing test
   - Given f is differentiable on I, if $f' > 0$ over I, f is increasing; if $f' < 0$, f is decreasing; else if $f' = 0$, f is constant
   - Proof: Using the MVT
     - Consider any $x_1 < x_2 \in I$, since f is differentiable, the MVT holds
     - Therefore there is some $c \in I$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, but as f' is positive over I, $f(x_2) > f(x_1)$, so by definition f is increasing
     - Proof is similar for the two other cases
2. First Derivative Test
   - Since $f(c_{crit})$ includes maxima, minima, and other values, we want to be able to know what values to keep
   - Given I contains a critical point c, f continuous at c, and f differentiable in I but not necessarily at c:
     1. If $f' > 0$ just to the left of c and $f' < 0$ just to the right of c, then c is a local maximum
     2. If $f' < 0$ just to the left of c and $f' > 0$ just to the right of c, then c is a local minimum
     3. If f' does not change sign, then c is neither a maximum nor a minimum
   - Proof of (1):
     - The statement of (1) means that there is some a such that $f' > 0$ for $x \in (a, c)$, so by quick test 1 f is increasing on (a, c)
     - Therefore $f(c) \geq f(x)$ for all $x \in (a, c)$
     - By the same logic there is some b such that $f(c) \geq f(b)$ for all $x \in (c, b)$
     - Therefore $f(c) \geq f(x)$ for $x \in (a, b)$, which by definition is a local maximum
   - (2) and (3) could be proven similarly
   - Note that continuity is strictly required
3. Concavity Test (for definition refer to next section)
   - Given $f(x)$ is twice differentiable on I, $f'' > 0 \implies f$ is concave up; $f'' < 0 \implies f$ is concave down
   - Proof: Textbook A46
     - Suppose a is the point of interest and $f''(a) > 0$; we need to show that the function lies above the tangent, i.e. $f(x) > f(a) + f'(a)(x - a)$ for $x \in I$ and $x \neq a$
     - Suppose $x > a$:

        * Applying the MVT we have $f(x) - f(a) = f'(c)(x-a)$ where $c \in (a, x)$
        * Since $f'' > 0$ we know f' is increasing on this interval, so $f'(a) < f'(c) \implies f'(a)(x-a) < f'(c)(x-a) \implies f(a) + f'(a)(x-a) < f(a) + f'(c)(x-a)$
        * But $f(x) = f(a) + f'(c)(x-a)$ because of the MVT, so $f(a) + f'(a)(x-a) < f(x)$, therefore the function lies above the tangent and it is concave up
        * Other cases can be proved similarly
4. Second Derivative Test
    - Given f'' exists and is continuous near c, and $f'(c) = 0$ then if $f''(c) > 0$ then $f(c)$ is a local minimum; if $f''(c) < 0$ then $f(c)$ is a local maximum; if $f''(c) = 0$ this test is inconclusive

Concavity and Points of Inflection

- Definition: If the graph of $y = f(x)$ lies above all its tangents in I, then $f(x)$ is concave up in I; similarly if it lies below all its tangents then $f(x)$ is concave down
- Definition: $f(x)$ has a point of inflection at c if $f(x)$ is continuous at c and the sign of concavity (up/down) changes at c