Lecture 13, Oct 8, 2021

The Mean Value Theorem

- *Rolle's Theorem*: Given $f(x)$ continuous on $[a, b]$, differentiable on $[a, b)$, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$ (there may be more than one)
  1. $f(x) > f(a) = f(b)$ for some $x \in (a, b)$ (i.e. the value at x is greater than the value at the end points a and b)
     - By the Extreme Value Theorem there exists an absolute maximum $f(c)$ somewhere in $[a, b]$, which cannot be an endpoint max (because $f(x) > f(a)$)
     - Therefore the maximum must be a local maximum
     - Then by Fermat's theorem c is a critical point
     - $f(c)$ cannot be undefined because $f(x)$ is differentiable on $[a, b)$ so the derivative exists for all values in $[a, b)$
     - Therefore $f'(c) = 0$
  2. $f(x) < f(a) = f(b)$ for some $x \in (a, b)$
     - Similar to (1), there exists a minimum which is not an endpoint minimum so it is a local minimum; therefore it is a critical point; the derivative at this minimum cannot be undefined, therefore the value of f' at this critical point must be 0
  3. $f(x) = f(a) = f(b)$ (constant)
     - Since $f(x)$ is a constant then $f'(x) = 0$ by the constant derivative theorem
- The *Mean Value Theorem* is a generalization of Rolle's theorem: Given $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, i.e. the derivative at c equals the mean rate of change of f over $[a, b]$
  - Define $y(x)$ to be the secant line from $(a, f(a))$ to $(b, f(b))$ and $g(x) \equiv f(x) - y(x)$
  - Then $g(x)$ is continuous over $[a, b]$ because both $f(x)$ is continuous (as given) and $y(x)$ is continuous (by the polynomial continuity theorem)
  - $g(x)$ is differentiable over (a, b) because both $f(x)$ is differentiable (as given) and $y(x)$ is differentiable (by the polynomial derivative theorem)
  - $g(a) = g(b) = 0$ since at a and b the secant line intersects $f(x)$
  - Therefore $g(x)$ meets the requirements for Rolle's Theorem and so there exists some $c \in (a, b)$ such that $g'(c) = 0$
  - $g'(c) = 0 \implies f'(c) - y'(c) = 0 \implies f'(c) = y'(c) = \dfrac{f(b) - f(a)}{b - a}$

The Structure of Calculus Logic

- Define numbers with the field and order axioms $\to$rationals $\to$functions $\to$limits $\to$
  - Continuity
  - Derivatives
- CORA $\to$irrationals $\xrightarrow{\text{+continuity}}$ IVT, EVT $\xrightarrow{\text{+derivatives}}$ MVT

Differentials

- Suppose $y = f(x)$; define two new variables Δy and Δx related by $\Delta y \equiv f(x + \Delta x) - f(x)$
  - Note the value of Δy depends on **both** Δx and x
- Define *differentials* $\mathrm{d}x$ and $\mathrm{d}y$ related by $\mathrm{d}y \equiv f'(x)\,\mathrm{d}x$
- Note that since $\mathrm{d}x$ and Δx are both free variables, we can set $\mathrm{d}x = \Delta x$
  - Then Δy is the true value and $\mathrm{d}y$ is a tangent-line approximation of f which improves as $\Delta x \to 0$
  - Practical note: Sometimes we want Δy but it might not be straightforward to compute it, so we can use $\mathrm{d}y$ as an approximation if Δx is small
- Note: The $\mathrm{d}x$ and $\mathrm{d}y$ here are not the same as the ones in $\dfrac{\mathrm{d}y}{\mathrm{d}x}$; the derivative is not a fraction, but the $\mathrm{d}y$ and $\mathrm{d}x$ here *are* numbers; it's just bad notation