Lecture 12, Oct 6, 2021

Applications of Derivatives

- Definition: $f(x)$ has an *absolute* maximum at c if $f(c) \geq f(x)$ for all $x \in$ domain $f(x)$
  - **Note $f(x)$ must exist!**
- $f(x)$ has an absolute maximum at c in $[a, b]$ if $f(c) \geq f(x)$ for all $x \in [a, b]$
- $f(x)$ has a local maximum at c if $f(c) \geq f(x)$ for some open interval containing c
  - This interval may be very small but it has to exist on both sides of c

Extreme Value Theorem

- Given $f(x)$ continuous on $[a, b]$, $f(x)$ has an absolute maximum $f(c)$ and an absolute minimum $f(d)$ for some $c, d \in [a, b]$
  - Knowing whether there exists a maximum/minimum is very important because finding maxima/minima is hard!
  - Proof: Supplement pages 52-56
  - Being defined does not imply boundedness; e.g. $\begin{cases} \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is defined everywhere but unbounded
    * Only continuity implies boundedness
  - Part 1: Continuity on an interval implies boundedness
    * Lemma 1: Given $f(x)$ is defined and continuous on $[a, b]$ and $a < c < b$, there exists some interval $(c - \delta, c + \delta)$ with $\delta > 0$, in $[a, b]$ for which $f(x)$ is bounded
      - This can be easily proven using continuity and epsilon delta; taking $\varepsilon = 1$ implies that on some interval $c - \delta < x < c + \delta$, $|f(x) - f(c)| < 1$
    * Lemma 2: Given $f(x)$ is defined and continuous on $[a, b]$, $f(x)$ is bounded on $[a, b]$
      - Let S be the set of all u for which $f(x)$ is bounded on $[a, u]$
      - Using Lemma 1, $f(x)$ is bounded on some interval $[a, u]$ with $u > a$, so S is nonempty; as $u < b$, S is bounded above by b and so by CORA has a least upper bound c
      - Suppose $c < b$, then by Lemma 1 there exists $\delta > 0$ such that $f(x)$ is bounded on $(c - \delta, c + \delta)$; therefore, $f(x)$ is bounded on $[a, c + \delta)$, which contradicts the statement that c is the least upper bound of S (since if this were true, the least upper bound of S would be $c + \delta$); therefore $c = b$
      - Now we know that $f(x)$ is bounded on $[a, b)$ (b is not closed because the least upper bound is not necessarily a member of the set)
      - Since f is continuous at b, it is bounded at b; then by adapting Lemma 1 to an endpoint we know f is bounded on $(b - \delta, b]$; since $f(x)$ is bounded on both $[a, b)$ and $(b - \delta, b]$, $f(x)$ is bounded on $[a, b]$
  - Part 2: Continuity on an interval implies the existence of a maximum and minimum
    * Consider the set $S = \{f(x) : a \leq x \leq b\}$; then S is bounded above since $f(x)$ is bounded and therefore by CORA it has a least upper bound M, so $f(x) \leq M$ for all $x \in [a, b]$
    * Now we need to prove that $f(c) = M$ since a maximum requires that the function takes on that value
    * Suppose $f(x)$ never equals M, then we can define $g(x) \equiv \dfrac{1}{M - f(x)}$, so $g(x)$ is always positive and defined on $[a, b]$, so $g(x)$ is continuous by the additivity and quotient continuity theorems
      - Therefore, $g(x)$ is also bounded on $[a, b]$ by part 1, so there exists some number K such that $0 < g(x) \leq K$, and $K > 0$
      - $\dfrac{1}{K} \leq \dfrac{1}{g(x)} = M - f(x) \implies f(x) \leq M - \dfrac{1}{K}$, but this violates the fact that M is the least upper bound of S since $M - \dfrac{1}{K}$ is smaller
    * Therefore, by contradiction, there is some $c \in [a, b]$ such that $f(c) = M$
- Note that being bounded is not the same as having an absolute maximum/minimum; e.g. $\dfrac{\sin x}{x}$ is

bounded but does not have a maximum

  - $\frac{\sin x}{x}$ is bounded by 1, but we cannot say that the maximum is 1, because it is undefined at 0 and so never takes on the value of 1

Fermat's Theorem

- Definition: c is a critical point of $f(x)$ if $f'(c) = 0$ or DNE
- Fermat's Theorem: Given $f(x)$ has a local maximum or minimum at c, then c is a critical point
  - The reverse is not true! $f'(c) = 0$ or DNE does not always imply that c is a local minimum or maximum
  - Note: This does not apply for maximum or minimum at end points of a range
  - Proof: textbook page 212
    * Suppose $f(c)$ is a local maximum; then $f(c) \geq f(x)$ or $f(c) \geq f(c+h)$ for some small h, positive or negative, so $f(c+h) - f(c) \leq 0$
    * Suppose h is positive, then:
      - $\frac{f(c+h) - f(c)}{h} \leq 0 \implies \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0 \implies f'(c) \leq 0$
    * Suppose h is negative, then:
      - $\frac{f(c+h) - f(c)}{h} \geq 0 \implies \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \geq 0 \implies f'(c) \geq 0$
      - Note here the direction of the inequality is flipped since we divided by h, a negative quantity
    * Since both $f'(c) \geq 0$ and $f'(c) \leq 0$, $f'(c) = 0$ must be true if it exists; alternatively it may be undefined
- Given a continuous $f(x)$ on $[a, b]$, by the extreme value theorem an absolute maximum/minimum of the range must exist; to test for the absolute maximum/minimum:
  1. Find all critical points c and $f(c)$
  2. Find the endpoints $f(a)$, $f(b)$
  3. The largest f value is the absolute maximum, the minimum f value is the absolute minimum