Lecture 11, Oct 4, 2021

Derivatives of Trig Functions

- In cases of $\frac{0}{0}$ our intuition fails
- Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{0} = 0$
  - We can't use the product limit theorem because $\lim_{x\to 0} \frac{1}{x}$ does not exist

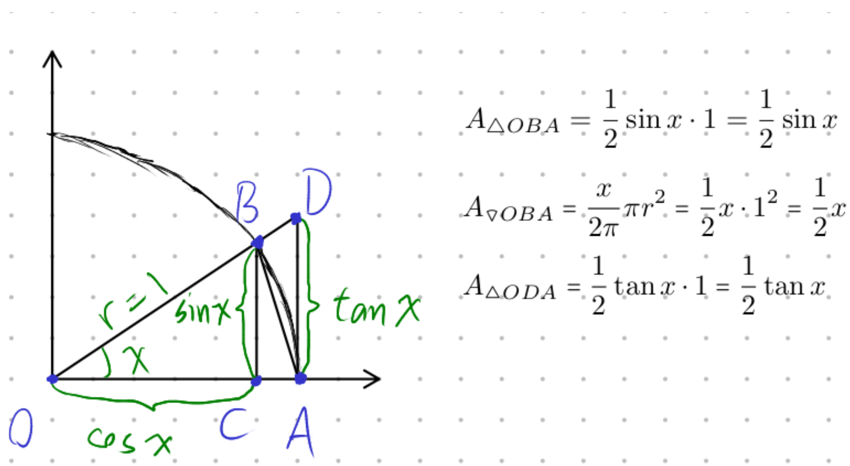

Figure 1: geometric proof

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  - $A_{\triangle OBA} \leq A_{OBA} \leq A_{\triangle ODA} \implies \sin x \leq x \leq \tan x \implies 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \implies \cos x \leq \frac{\sin x}{x} \leq 1$
  - We can use the sandwich theorem since $\cos 0 = 1 \implies \lim_{x\to 0} \frac{\sin x}{x} = 1$
  - $$\begin{aligned} \lim_{x\to 0} \frac{\cos x - 1}{x} &= \lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} \\ &= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\ &= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} \\ &= 1 \cdot \frac{\lim_{x\to 0} -\sin x}{\lim_{x\to 0} \cos x + 1} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

- $\displaystyle \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$
$\displaystyle = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$
$\displaystyle = \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$
$\displaystyle = \lim_{h\to 0}\sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\lim_{h\to 0}\frac{\sin h}{h}$
$= 0\cdot\sin x+1\cdot\cos x$
$= \cos x$
- $\displaystyle \frac{\mathrm{d}}{\mathrm{d}x}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$
$\displaystyle = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$
$\displaystyle = \lim_{h\to 0}\frac{\cos x(\cos h-1)+\sin x\sin h}{h}$
$\displaystyle = \lim_{h\to 0}\cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\lim_{h\to 0}\sin x\lim_{h\to 0}\frac{\sin h}{h}$
$= 0\cdot\cos x-1\cdot\sin x$
$= -\sin x$

Chain Rule (Composite Function Derivative Theorem)

- The chain rule theorem: $(f(u(x)))' = f'(u)u'(x)$ or $\displaystyle \frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}f}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$
  - Simple proof, assuming $u(x)\neq u(a)$ when x and a are close:
    * $\displaystyle (f(u(x)))' = \lim_{a\to x}\frac{f(u(a))-f(u(x))}{a-x}$
    $\displaystyle = \lim_{a\to x}\frac{f(u(a))-f(u(x))}{u(a)-u(x)}\lim_{a\to x}\frac{u(a)-u(x)}{a-x}$
    $\displaystyle = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$

Implicit Differentiation

- Sometimes there is no explicit expression for $y(x)$ and we only have an implicit relation, e.g. $x^3y^7 - x^2+y^2=0$
- We can find $y'(x)$ here using implicit differentiation and apply the "$\frac{\mathrm{d}}{\mathrm{d}x}$ operator" to both sides
- Using implicit differentiation we can prove that the power derivative theorem works for any rational number
  - Set $u(x)\equiv x^{\frac{p}{q}}$ where p and q are integers; note $u^q = x^p$
  - Set $f(u)\equiv u^q = x^p$; note $f(u(x))$ is a composite function
  - $\displaystyle \frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \implies px^{p-1}=qu^{q-1}\frac{\mathrm{d}u}{\mathrm{d}x} \implies u'=\frac{px^{p-1}}{qu^{q-1}}$
  - $u = x^{\frac{p}{q}} \implies u^{q-1}=x^{\frac{p}{q}(q-1)}=x^{p-\frac{p}{q}}$
  - $\displaystyle u'=\frac{px^{p-1}}{qu^{q-1}}=\frac{p}{q}\frac{x^{p-1}}{x^{p-\frac{p}{q}}}=\frac{p}{q}x^{\frac{p}{q}-1}$

Higher Derivatives

- Assuming $s'(t)$ exists, it is a function itself, so we can also take its derivative
- $s'(t)$ is the *first derivative*, $s''(t)$ is the *second derivative*, the derivative of the derivative, and so on

- In Newtonian notation $(s'(t))' = s''(t)$, in Leibniz $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ and so on for higher orders
    - With Leibniz we can use unit checking to check for potential mistakes; by putting the exponent on t but not s in $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ we know the units for t are squared but the units for s are not