Lecture 10, Oct 1, 2021

Differentiability

- Definition: $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable for all $x \in (a, b)$
- Definition: $f(x)$ is differentiable on $[a, b]$ if:
  1. $f(x)$ is differentiable on (a, b)
  2. The right hand derivative exists at a
  3. The left hand derivative exists at b
- The left hand derivative is defined as $f'(x^-) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$
- The right hand derivative is the same except the limit is now $h \to 0^+$
- Example: $f(x) = |x|$ is not differentiable at 0, but both the left and right hand derivative exist (but do not equal)
- $f(x)$ may be continuous at a point, but not differentiable; differentiability is "rarer" than continuity
- Differentiability implies continuity:
  - Since continuity guarantees integrability, differentiability guarantees integrability
  - $$\begin{aligned} &\lim_{h \to 0} [f(a+h) - f(a)] \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} h \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \lim_{h \to 0} h \\ &= f'(a) \cdot 0 \\ &= 0 \end{aligned}$$
  - $$\begin{aligned} &\lim_{h \to 0} [f(a+h) - f(a)] = 0 \\ \implies &\lim_{h \to 0} f(a+h) - \lim_{h \to 0} f(a) = 0 \\ \implies &\lim_{h \to 0} f(a+h) = f(a) \\ \implies &\lim_{(x-a) \to 0} f(x) = f(a) \text{ substitute } x = a + h \\ \implies &\lim_{x \to a} f(x) = f(a) \\ \implies &f \text{ is continuous at } a \end{aligned}$$
  - Note the reverse is not true! Not being differentiable does not imply discontinuity

Vertical Tangent Lines

- Example: $f(x) = \sqrt[3]{x} \implies f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \implies \lim_{x \to 0} |f'(x)| = \infty$
  - Note that $f(x)$ is **not** differentiable at c
- $f(x)$ has a vertical tangent at c if:
  1. $\lim_{x \to c} |f'(x)| = \infty$
  2. $f(x)$ is continuous at c
     - Vertical tangents are different from vertical asymptotes and this separates them
     - e.g. $f(x) = \frac{1}{x^2} \implies f'(x) = -\frac{2}{x^3}$ which is infinite at 0, but this is not a vertical tangent since $f(x)$ is not continuous at this point

Derivative Theorems

- Just like limit theorems, these allow us to compute derivatives without having to evaluate limits
- Some theorems are outlined here:
  1. Constant Derivative Theorem: $f(x) = C \implies f'(x) = 0$
     - Proof is trivial

2. Additive Derivative Theorem: $(f+g)' = f' + g'$ **if both exist**
   - Proof is trivial
3. Product Derivative Theorem: $(fg)' = f'g + fg'$
   - $$\begin{aligned}&(fg)' \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h} \\ &= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x)\end{aligned}$$
4. Power Derivative Theorem: $f(x) = Cx^n \implies f'(x) = nCx^{n-1}$
5. Polynomial Derivative Theorem
6. Reciprocal function Derivative Theorem: $\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}$ for $f(x) \neq 0$
   - $$\begin{aligned}&\left(\frac{1}{f(x)}\right)' \\ &= \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} \\ &= \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x)f(x+h} \\ &= \lim_{h\to 0} \frac{f(x) - f(x+h)}{h} \lim_{h\to 0} \frac{1}{f(x)} \lim_{h\to 0} \frac{1}{f(x+h)} \\ &= -f'(x)\frac{1}{f(x)}\frac{1}{f(x)} \\ &= -\frac{f'(x)}{f^2(x)}\end{aligned}$$
7. Quotient Derivative Theorem: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ for $g(x) \neq 0$
   - Use product and power derivative theorems to prove

- Using ⑥ we can prove ④ works for negative powers:
  - $f(x) \equiv x^{-n}$ where n is a positive integer
  - $g(x) \equiv x^n \implies f(x) = \frac{1}{g(x)}$
  - $f'(x) = -\frac{g'(x)}{g^2(x)} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$