Lecture 1, Sep 13, 2021

Defining the Derivative

- The start of this course involves defining the derivative *rigorously and logically*
- We can't simply define it as the slope of a tangent line or instantaneous speed because neither are well-defined
- Example: Consider a falling object with distance covered $d(t) = 5t^2$, which is an algebraic approximation that comes from experiments; what is the instantaneous speed at $t = 1$?
  - Consider the secant line between $d(t)$ and $d(t + x)$: $f(x) \equiv \dfrac{d(1+x) - d(1)}{x} = \dfrac{5(1+x)^2 - 5(1)}{x} = \dfrac{5 + 10x + 5x^2 - 5}{x} = \dfrac{10x + 5x^2}{x} = (10 + 5x)\left(\dfrac{x}{x}\right)$
  - If $\dfrac{x}{x} = 1$ for all x then the instantaneous speed is $f(0) = 1$; however this is **not true** since division by zero is undefined

Defining The Limit

- The limit is invented by mathematicians to get around division by zero
- Define $\dfrac{a}{b}$ as $a\dfrac{1}{b}$; define $\dfrac{1}{b}$ as the number where $b \cdot \dfrac{1}{b} = 1$
  - This definition excludes $b = 0$ because (theorem) 0 times any number is 0
  - We could choose to include $b = 0$ but in that case $b \cdot \dfrac{1}{b} = 0$ and so $0 = 1$ and all numbers collapse to 0, which is not very useful or fun
- Intuitive definition of $\lim\limits_{x \to a} f(x)$: The number that $f(x)$ gets closer and closer to when x gets closer and closer to a
  - From a rigorous point of view, this is not very useful
- Start by defining a simpler type of limit: e.g. $g(x) = 2 + \dfrac{1}{x^2}$ and $\lim\limits_{x \to \infty} g(x)$
  - The limit exists if we can always find some x large enough such that $g(x)$ is arbitrarily close to 2, e.g. within $\pm 10^{-10}$; for all $x > x_0$, $g(x)$ is within 10^{-10} of 2
  - We can try some big x_0, e.g. 10^{100}, then $x > x_0 \implies \dfrac{1}{x^2} < 10^{-200} \implies g(x) = 3 + \dfrac{1}{x^2} < 3 + 10^{-10}$
    * The other side of the inequality is satisfied by $x^2 \geq 0 \implies \dfrac{1}{x^2} \geq 0 \implies g(x) > 3 > 3 - 10^{-10}$
  - 10^{-10} was arbitrary; we can generalize this: the limit is 2 if for any small $\varepsilon > 0$ there exists an x_0 such that for all $x > x_0$, $g(x)$ is within ε of 2
    * Take $x_0 = \dfrac{1}{\sqrt{\varepsilon}} \implies x > \dfrac{1}{\sqrt{\varepsilon}} \implies x^2 > \dfrac{1}{\varepsilon} \implies \dfrac{1}{x^2} < \varepsilon \implies g(x) < 3 + \varepsilon$
  - This is the *guess-and-test* method: assume the limit exists, guess a value, and test it against ε