Lecture 1, Sep 13, 2021

Defining the Derivative

- The start of this course involves defining the derivative *rigorously and logically*
- We can't simply define it as the slope of a tangent line or instantaneous speed because neither are well-defined
- Example: Consider a falling object with distance covered $d(t) = 5t^2$, which is an algebraic approximation that comes from experiments; what is the instantaneous speed at $t = 1$?
  - Consider the secant line between $d(t)$ and $d(t+x)$: $f(x) \equiv \frac{d(1+x) - d(1)}{x} = \frac{5(1+x)^2 - 5(1)}{x} = \frac{5 + 10x + 5x^2 - 5}{x} = \frac{10x + 5x^2}{x} = (10 + 5x)\left(\frac{x}{x}\right)$
  - If $\frac{x}{x} = 1$ for all x then the instantaneous speed is $f(0) = 1$; however this is **not true** since division by zero is undefined

Defining The Limit

- The limit is invented by mathematicians to get around division by zero
- Define $\frac{a}{b}$ as $a\frac{1}{b}$; define $\frac{1}{b}$ as the number where $b \cdot \frac{1}{b} = 1$
  - This definition excludes $b = 0$ because (theorem) 0 times any number is 0
  - We could choose to include $b = 0$ but in that case $b \cdot \frac{1}{b} = 0$ and so $0 = 1$ and all numbers collapse to 0, which is not very useful or fun
- Intuitive definition of $\lim_{x \to a} f(x)$: The number that $f(x)$ gets closer and closer to when x gets closer and closer to a
  - From a rigorous point of view, this is not very useful
- Start by defining a simpler type of limit: e.g. $g(x) = 2 + \frac{1}{x^2}$ and $\lim_{x \to \infty} g(x)$
  - The limit exists if we can always find some x large enough such that $g(x)$ is arbitrarily close to 2, e.g. within $\pm 10^{-10}$; for all $x > x_0$, $g(x)$ is within 10^{-10} of 2
  - We can try some big x_0, e.g. 10^{100}, then $x > x_0 \implies \frac{1}{x^2} < 10^{-200} \implies g(x) = 3 + \frac{1}{x^2} < 3 + 10^{-10}$
    - The other side of the inequality is satisfied by $x^2 \geq 0 \implies \frac{1}{x^2} \geq 0 \implies g(x) > 3 > 3 - 10^{-10}$
  - 10^{-10} was arbitrary; we can generalize this: the limit is 2 if for any small $\varepsilon > 0$ there exists an x_0 such that for all $x > x_0$, $g(x)$ is within ε of 2
    - Take $x_0 = \frac{1}{\sqrt{\varepsilon}} \implies x > \frac{1}{\sqrt{\varepsilon}} \implies x^2 > \frac{1}{\varepsilon} \implies \frac{1}{x^2} < \varepsilon \implies g(x) < 3 + \varepsilon$
  - This is the *guess-and-test* method: assume the limit exists, guess a value, and test it against ε

Lecture 2 (Online)

Defining Numbers

- Numbers are the most basic elements and math and thus can't be defined explicitly
- Numbers are defined implicitly by imposing the **axioms**, basic rules, that they have to satisfy
- Some axioms:
  1. Commutative law $x + y = y + x$
  2. Associative law $(x + y) + z = x + (y + z)$
  3. Distributive law $x(y + z) = xy + xz$
  4. Existence of additive identity 0 and multiplicative identity 1
  5. Existence of additive inverses $-x$ and multiplicative inverses x^{-1} for $x \neq 0$
- There should be as few axioms as possible as axioms are unprovable; the more axioms, the more the risk of contradiction

- Theorems arise from definitions and axioms and are provable: Example: $4 = 2 + 2$; there is no limit to the number of theorems
- Some definitions:
  1. Positive integers ("natural numbers"): $1, 2, 3, \ldots$; $2 \equiv 1 + 1$
  2. Rational numbers: $\dfrac{a}{b} \equiv a \cdot \dfrac{1}{b}$ where a and b are integers and $b \neq 0$
     - The axiom for the existence of inverses combined with this creates rational numbers
  3. Roots, other algebraic (root of a polynomial) irrational numbers; e.g. $q^2 - 2 = 0$
     - Axiom 5 only creates rational numbers, so a new one is needed for irrational numbers
  4. Transcendental irrationals; e.g. π or e
- Axiom: *Completeness of the Reals Axiom* (CORA): Every non-empty set of real numbers that is bounded above has a least upper bound among the real numbers; this creates irrationals
  - A set of real numbers S_1 is bounded above iff there exists some ubS_1 such that $x \leq \mathrm{ubS}_1$ for all $x \in S_1$; the least upper bound lubS_1 is the least of the ubS_1
    * The least upper bound does not have to be in S
  - Example: $S_2 = \{x : x^2 < 2\}$
    * lubS_2 intuitively would satisfy $\mathrm{lubS}_2^2 = 2$ and so $\mathrm{lubS}_2 = \sqrt{2}$
    * By imposing CORA, $\sqrt{2}$ and all irrationals (including transcendental irrationals) have been created
    * However proving $\mathrm{lubS}_2 = \sqrt{2}$ rigorously is very difficult

Lecture 3, Sep 15, 2021

Review

1. Absolute value $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$
   - Easier to work with: $|a| = \sqrt{a^2}$
     - Note: Square roots are zero or positive or does not exist in the reals; e.g. $\sqrt{4}$ is **only** 2, not ± 2
     - Different question: What x satisfies $x^2 - 4 = 0$? This is a *different question* and is ± 2
2. Intervals of x values
   1. $x \in [a, b]$ – closed interval ($a \leq x \leq b$), which is a set of numbers; filled dots on the number line
   2. $x \in (a, b)$ – open interval ($a < x < b$); open dots on the number line
   3. $x \in [a, b)$, $x \in (a, b]$ – mixed open/close intervals
   4. $x \in (-\infty, b]$ – all $x \leq b$
      - Infinity is only okay to use inside an expression when the whole expression is defined, because ∞ is not a number
      - Closed brackets on infinity, e.g. $[a, \infty]$, is undefined because there is no useful definition for it
3. Functions: Given two sets of numbers x-set and y-set, a function is a rule that we specify that relates each x to *one* y value (very general)
   - x *independent* variable, y *dependent* variable; x could be anything as long as it produces a defined y
   - Note the asymmetry: each x can only relate to one y, but this requirement does not exist for y
   - Example: A table of x and y values could be a function
   - The span of the set of x values is the *domain*; the span of the set of y values is the *range*
   - Functions do not have to be **well-behaved**: e.g. $f(x) = \begin{cases} 10x + 5 & x \neq 0 \\ DNE & x = 0 \end{cases}$ is a perfectly good function; so is $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$
4. Trigonometric functions:
   - Prefer algebraic definitions over geometric definitions
   - Prefer angles to be in radians where 2πrad is a full revolution (geometric definition of π)
     - For radius r the arc length corresponding to angle x is just rx (check: if $x = 2\pi$ then $rx = 2\pi r$)

5. Composition of functions: $f(g(x))$
   - e.g. if $f(x) = 3x^2 + 2$ and $g(x) = \sin x$ then $f(g(x)) = 3\sin^2 x + 2$ and $g(f(x)) = \sin(3x^2 + 2)$
6. Increasing/decreasing functions
   - Given $x_1 > x_2$ for any two values of x_1 and x_2 in some interval, if $f(x_1) > f(x_2)$, define $f(x)$ to be **increasing** on this interval; if $f(x_1) < f(x_2)$, define $f(x)$ to be **decreasing** on this interval
   - Example: Prove $x^2 + 2$ is increasing for $x > 0$
     - Take any $x_1 > x_2 > 0$
     - $0 < x_1^2 < x_1 x_2$ and $0 < x_1 x_2 < x_2^2 \implies x_1^2 < x_2^2 \implies x_1^2 + 2 < x_2^2 + 2 \implies f(x_1) < f(x_2)$
7. Even/odd functions
   - $f(x) = f(-x)$ is an even function; $f(x) = -f(-x)$ is an odd function
8. Basic Arithmetic Theorem (BAT1)
   - Prove a true equality: e.g. $1 + 2 = 6 - 3$
     - You can add, subtract, multiply, and divide both sides of an equality by the same factor and get another true equality
     - This is a theorem and can be proven
   - Prove a true inequality: e.g. $3 < 5$
     - Get another true inequality by applying the same operations, **except** if multiplication or division is involved *and* the factor is negative, then the direction of the inequality changes
     - Also a theorem that can be proved using axioms

Lecture 4, Sep 17, 2021

Review, Continued

- BAT2 (Basic Algebraic Theorem)
  - Note: $1 + 2 = 6 - 3$ is a *statement* and therefore can be true or false
  - $x + 3 = 7x - 1$ is **not** a statement (not meaningful to ask whether it is true or false); it is a *prescription* for the value of x; $x^2 - 4 = 0$ is a prescription for 2 values of x
  - Similarly $3 < 5$ is a statement (true/false); $x < 4$ is a prescription for a whole set of x values
  - BAT2 is equal to BAT1 except the factor could now be algebraic; since the sign might be hidden inequalities require more work
  - e.g. Prove $f(x) = x^2 + 3$ is decreasing for $x < 0$
    * Pick any $x_1 < x_2 < 0$
    * $x_1 < x_2 \implies x_1^2 > x_1 x_2$ **note the direction change because** $x_1 < 0$**!**
    * $x_1 < x_2 \implies x_1 x_2 > x_2^2$ same direction change here because $x_2 < 0$
    * See definition of increasing/decreasing functions from Lecture 3 for rest of proof
  - $x^2 - x - 6 > 0 \implies (x - 3)(x + 2) > 0$
    * First subset: $x - 3 > 0$ and $x + 2 > 0$ **for the same** x
    * Second subset: $x - 3 < 0$ and $x + 2 < 0$ **for the same** x
  - Equalities prescribe a number of x values; inequalities prescribe a set of x values (or none at all if the inequality is impossible)
- Absolute value functions $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$
  - Example: $f(x) = |x + 3| = \begin{cases} x + 3 & x + 3 \geq 0 \\ -x - 3 & x + 3 < 0 \end{cases}$
  - Example: What values of x satisfy $|x + 3| = 5$?
    * Possibility 1: $x + 3 > 0 \implies x + 3 = 5$ so $x > -3$ and at the same time $x = 2$, which is possible
    * Possibility 2: $x + 3 < 0 \implies -x - 3 = 5$ so $x < -3$ and $x = -8$, which is also possible
  - Example: $|x + 3| < 5$
    * Possibility 1: $x + 3 \geq 0 \implies x + 3 < 5$ so $x \geq -3$ and $x < 2$, so $-3 \leq x < 2$
    * Possibility 2: $x + 3 < 0 \implies -x - 3 < 5$ so $x < -3$ and $x > -8$, so $-8 < x < -3$
    * Combine the two sets: $-8 < x < 2$

    * -3 is halfway between 2 and -8 with half-width 5 (generalizes to other inequalities of the same form)
        - Example: $|x-c| < \delta$ where c can be positive or negative and $\delta > 0$; $+c$ is the center and δ is the half-width: $c - \delta < x < c + \delta$
  - The set $0 < |x-c| < \delta$ excludes $x = c$; this is important in the context of limits since the function may be undefined at c
    * If $0 < |f(x) - c| < \delta$ says that $f(x)$ can never go above or below $c \pm \delta$, so this bounds its range, useful for limits

Lecture 5, Sep 20, 2021

Rigorous Definition of the Limit

- Test-definition (a type of implicit definition) for a new number $\lim_{x \to c} f(x)$, given:
  1. c, some particular value of x
  2. $f(x)$, which may be undefined at c, but is defined for all x near c
  3. L, a candidate value for the limit
- Imposed: Some small positive $\varepsilon > 0$; we don't have the exact value and will have to allow for any $\varepsilon > 0$
- Test: Find some $\delta > 0$, such that for all $0 < |x-c| < \delta$, $|f(x) - L| < \varepsilon$
  - i.e. Find some δ such that all x within the x-band have corresponding values of f that fall in the y-band
- If the test passes, then the limit exists and $\lim_{x \to c} f(x) = L$
- Since $|x-c| > 0$, x is never really equal to c, so we can simplify situations such as $\frac{x}{x}$ when $c = 0$ legitimately

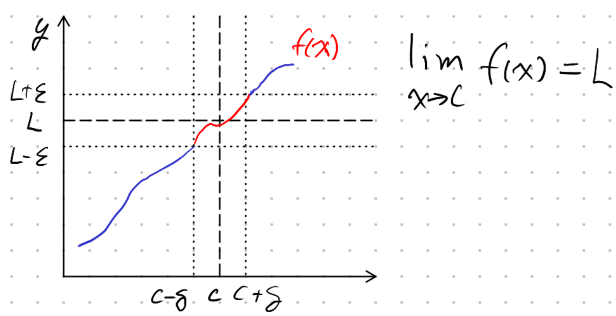


Figure 1: graph illustration

General Process

1. $\varepsilon > 0$ is imposed (it is **given** so we can and have to work with it)
2. Find a set of x values for which $|f(x) - L| < \varepsilon$
   - Example: Prove $\lim_{x \to 0} \frac{10x + 5x^2}{x} = 10$
   - Here $\left| \frac{10x + 5x^2}{x} - 10 \right| = |10 - 5x - 10| = |5x| = 5|x| < \varepsilon$
   - Notice how the x can be cancelled out rigorously now since x is not allowed to be zero
3. Look for a set of x values you will specify by $0 < |x - c| < \delta$
   - Example: $0 < |x - 0| = |x| < \delta$
4. Plug ③ into the left hand side of ② and do algebraic manipulation until you get $|f(x) - L| <$ some expression involving only δ
   - Example: If $5|x| < \varepsilon$ for ② and $|x| < \delta$, then $5|x| < 5\delta$

5. Guess δ in terms of ε and plug back in to get $|f(x) - L| <$ some expression involving ε, and then make the right hand side $< \varepsilon$
   - Example: Choose $\delta = \frac{1}{5}\varepsilon$, substitute into $5|x| < 5 \implies 5|x| < \varepsilon$
   - Now we've found (one of the) δ values for any given ε such that $|f(x) < L| < \varepsilon$ for all $0 < |x-c| < \delta$, so we can conclude $\lim_{x \to c} f(x) = L$
6. Compact: Given $\varepsilon > 0$, take $\delta = \cdots$ then when $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$, therefore $\lim_{x \to c} f(x) = L$

Example

- Prove $\lim_{x \to 0} x^3 = 0$
  - $|x^3 - 0| = |x^3| = |x|^3 < \varepsilon$
  - $0 < |x - 0| = |x| < \delta$
  - $|x|^3 < \delta^3$
  - Take $\delta = \sqrt[3]{\varepsilon} \implies |x|^3 < \varepsilon$; QED
  - Note the choice of δ is not unique; anything that does the job is fine!

Lecture 6, Sep 21, 2021

- The minimum function lets us unambiguously prescribe multiple things about a variable
- Example: Prove $\lim_{x \to 5} x^2 = 25$
  1. $\varepsilon > 0$ is given
  2. Required $|f(x) - L| = |x^2 - 25| < \varepsilon$
  3. When $0 < |x - c| = |x - 5| < \delta$
  4. $|x^2 - 25| = |(x - 5)(x + 5)| = |x - 5||x + 5| < \delta|x + 5|$
     - Now we need to get rid of $|x + 5|$ by specifying an additional feature of δ
     - If we specify $\delta \leq 1 \implies |x-5| < \delta \leq 1 \implies 4 \leq x \leq 6 \implies 9 \leq x+5 \leq 11 \implies |x+5| \leq 11$
     - $|x + 5| \leq 11 \implies \delta|x + 5| \leq 11\delta \implies |x - 5||x + 5| < 11\delta$
  5. Now we can take $\delta = \frac{\varepsilon}{11} \implies |f(x) - L| = |x - 5||x + 5| < 11\delta = \varepsilon$
  6. Include constraint $\delta \leq 1$: Notice how anything smaller than $\frac{\varepsilon}{11}$ works, so we can say $\delta = \min\left(\frac{\varepsilon}{11}, 1\right)$

Left and Right-Hand Limits

- Consider $f(x) = \begin{cases} 1 + x^2 & x \geq 0 \\ x^2 & x < 0 \end{cases}$; in this case $\lim_{x \to 0} f(x)$ does not exist
- We can define the *left-hand limit*: $\lim_{x \to 0^-} f(x)$ and the *right-hand limit*: $\lim_{x \to 0^+} f(x)$
- The definition of the left and right hand limits are slight modifications to the normal limit:
  - Right-hand: $\lim_{x \to c^+} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0$ such that $\forall c < x < c + \delta, |f(x) - L| < \varepsilon$
  - Left-hand: $\lim_{x \to c^-} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0$ such that $\forall c - \delta < x < c, |f(x) - L| < \varepsilon$
  - We can then conclude that $\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$
- Example: Prove $\lim_{x \to 0^+} \sqrt{x} = 0$
  1. $\varepsilon > 0$ is given
  2. Required $|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$
  3. When $0 < x < \delta$
  4. $\sqrt{x} < \sqrt{\delta}$
  5. Take $\delta = \varepsilon^2 \implies |f(x) - L| = \sqrt{x} < \sqrt{\delta} = \varepsilon$

Vertical Asymptotes and Infinite Limits

- e.g. $f(x) = \frac{1}{x^4}$ goes to infinity at 0
- The definition of an infinite limit $\lim_{x \to c} f(x) = \infty \iff \forall M > 0, \exists \delta > 0$ such that $\forall 0 < |x - c| < \delta, f(x) > M$
  - Note that this is not an equation since ∞ is not a number!
  - $\lim_{x \to 0} \frac{1}{x^4}$ does not exist, but writing $\lim_{x \to c} f(x) = \infty$ is valid and tells us more than just saying it does not exists
  - This is just a shorthand of saying "$f(x)$ increases without limit as x approaches c"
- From this definition we can similarly construct definitions for limits of negative infinity and infinite one-handed limits
- Example: $\lim_{x \to 0} \frac{1}{x^4} = \infty$
  - $\frac{1}{x^4} > M$ when $0 < |x| < \delta$
  - $|x| < \delta \implies \frac{1}{\delta} < \frac{1}{|x|} \implies \frac{1}{\delta^4} < \frac{1}{|x|^4}$
  - Take $\delta = \frac{1}{M^{\frac{1}{4}}} \implies \frac{1}{x^4} = \frac{1}{|x|^4} < \frac{1}{\delta^4} = M$

Lecture 7, Sep 23, 2021

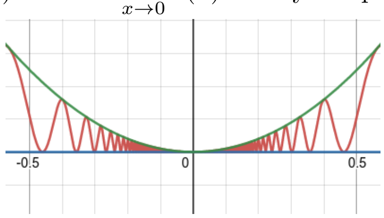
Limit Theorems

- Limit theorems let us rigorously prove much more complicated limits, such as polynomials, by breaking it into manageable pieces, each provable with the epsilon-delta definition
- Some limit theorems: (assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ are given; **both limits need to exist**)
  1. Additivity Limit Theorem: $\lim_{x \to c}[f(x) + g(x)] = L + M$
     - Proof:
       * Required $|f(x) + g(x) - L - M| < \varepsilon$ when $0 < |x - c| < \delta$
       * $|f(x) + g(x) - L - M| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|$ (triangle inequality)
       * We are given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, therefore:
         - For some $\varepsilon_f = \frac{\varepsilon}{2}$, there is $0 < |x - c| < \delta_f$ such that $|f(x) - L| < \varepsilon_f = \frac{\varepsilon}{2}$
         - For the same ε_f, there is $0 < |x - c| < \delta_g$ such that $|g(x) - M| < \varepsilon_f = \frac{\varepsilon}{2}$
       * If x is inside both δ_f and δ_g bands, then $|f(x)+g(x)-L-M| \leq |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
       * Therefore by picking $\delta = \min(\delta_f, \delta_g)$, $0 < |x - c| < \delta \implies |f(x) + g(x) - L - M| < \varepsilon$, and the limit is proved!
  2. Product Limit Theorem: $\lim_{x \to c} f(x)g(x) = LM$
     - Proof:
       * From the limits given: For some $\varepsilon_f = \sqrt{\varepsilon} > 0$ there is $0 < |x-c| < \delta_f \implies |(f(x)-L)-0| < \varepsilon_f$ and $0 < |x - c| < \delta_g \implies |(g(x) - M) - 0| < \varepsilon_g$
       * Therefore if $\delta = \min(\delta_f, \delta_g)$ then $0 < |x - c| < \delta \implies |(f(x) - L) - 0||(g(x) - M) - 0| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$, so $\lim_{x \to c}(f(x) - L)(g(x) - M) = 0$
       * $0 = \lim_{x \to c}(f(x) - L)(g(x) - M) = \lim_{x \to c}[f(x)g(x) - f(x)M - g(x)L + LM] = \lim_{x \to c} f(x)g(x) + \lim_{x \to c} -Mf(x) + \lim_{x \to c} -Lg(x) + \lim_{x \to c} LM$ by the additive theorem
       * Therefore $\lim_{x \to c} f(x)g(x) = -\lim_{x \to c} -Mf(x) - \lim_{x \to c} -Lg(x) - \lim_{x \to c} LM = ML + LM - LM = ML$ (by $\lim_{x \to c} cf(x) = c \lim_{x \to c} f(x)$, ~~proof of which is left as an exercise to the reader~~)

3. Polynomial Limit Theorem: $\lim_{x \to c} P_n(x) = P_n(c)$ for polynomials $P_n(x)$
   - This can be trivially proven using the product and additivity limit theorems
4. Rational Function Limit Theorem: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
   - Proof not included because I'm tired, see this link.
5. Root Limit Theorem: $\lim_{x \to c} \sqrt[n]{f(x)} = L^{\frac{1}{n}}$

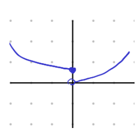
- Other limit theorems:
  6. Sandwich (aka Squeeze) Limit Theorem: If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$ and $f(x) \leq g(x) \leq h(x)$ near but not necessarily at c, then $\lim_{x \to c} g(x) = L$
     - $g(x)$ may be a very complicated function, but we might be able to find simple functions $f(x)$ and $h(x)$ that bound $g(x)$ near c for which we can easily find the limits of

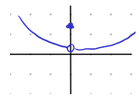
Applying Limit Theorems

- Example:
$$\begin{aligned} & \lim_{x \to -2} \frac{x^2 - x - 6}{x^2 - 4} \\ &= \lim_{x \to -2} \frac{(x-3)(x+2)}{(x-2)(x+2)} \\ &= \lim_{x \to -2} \frac{x-3}{x-2} \\ &= \frac{\lim_{x \to -2} x - 3}{\lim_{x \to -2} x - 2} \text{ (rational function LT)} \\ &= \frac{5}{4} \text{ (polynomial LT)} \end{aligned}$$
- Rigorously logical proofs justify every line by citing an axiom or proven theorem
- Example of using the Sandwich Theorem: $\lim_{x \to 0} x^2 \cos^2 \frac{1}{x^2}$
  - Can't use the product LT because $\lim_{x \to 0} \cos^2 \frac{1}{x^2}$ DNE (the function just oscillates faster and faster)
  - Find bounding functions: $0 \leq \cos^2 \frac{1}{x^2} \leq 1 \implies 0 \leq x^2 \cos^2 \frac{1}{x^2} \leq x^2$
  - Define $f(x) \equiv 0, g(x) \equiv x^2 \cos^2 \frac{1}{x^2}, h(x) \equiv x^2$
  - $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} h(x) = 0$ by the polynomial LT, therefore $\lim_{x \to 0} g(x) = 0$
  - Graph: 

Lecture 8, Sep 27, 2021

Continuity & Discontinuity

- The formal definition of continuity at a point is $f(x)$ is continuous at c iff $\lim_{x \to c} f(x) = f(c)$
- Types of discontinuity:
  1. Jump discontinuity: 

2. Removable discontinuity: 
3. $\lim_{x \to c}$ DNE or $f(c)$ DNE

Continuity Theorems

- Many important theorems hold true only for continuous functions, so it is important to be able to prove that functions are continuous
  - e.g. Continuity implies integrability; if $f(x)$ is continuous on $[a, b]$ then $\int_a^b f(x)\,\mathrm{d}x$ is guaranteed to exist (note: it might not be able to be expressed using already defined mathematical operations, but it must exist)
- There are continuity theorems just like limit theorems:
  1. Additivity Continuity Theorem: $f(x)$ and $g(x)$ continuous at $c \implies f(x) + g(x)$ continuous at c
     - Proofs are trivial using the corresponding limit theorems
  2. Product Continuity Theorem: $f(x)$ and $g(x)$ continuous at $c \implies f(x)g(x)$ continuous at c
  3. $\ldots$
- Composite Function Continuity Theorem: Given $g(x)$ is continuous at a and $f(x)$ is continuous at the number $g(a)$, then $f(g(x))$ is continuous
  - Using this we can test continuity for nested expressions, e.g. $f(x) = \tan\left(\frac{\pi x^2}{4}\right)$ at $x = 1$

One-Sided Continuity

- Just like limits, continuity can also be one-handed
- $f(x)$ is continuous on the left at c iff $\lim_{x \to c^-} f(x) = f(c)$; $f(x)$ is continuous on the right at c iff $\lim_{x \to c^+} f(x) = f(c)$
- e.g. $f(x) = \begin{cases} 1 + x^2 & x \geq 0 \\ x^2 & x < 0 \end{cases}$ is continuous from the right at 0 and discontinuous from the left at 0

Continuity On an Interval

- $f(x)$ is continuous on the open interval (a, b) iff $f(x)$ is continuous for all $x \in (a, b)$
- $f(x)$ is continuous on the closed interval $[a, b]$ iff $f(x)$ is continuous for all $x \in (a, b)$ **and** $f(x)$ is continuous from the right at a **and** $f(x)$ is continuous from the left at b
  - Notice that this doesn't require $f(x)$ to be continuous on both sides at a and b; just one-handed continuity is okay

Intermediate Value Theorem

- Given $f(x)$ continuous on $[a, b]$ and $f(a) < C < f(b)$ or $f(a) > C > f(b)$, there exists some $c \in [a, b]$ such that $f(c) = C$
- The point it that continuous functions don't "skip over" y values; without irrationals and transcendentals the IVT cannot be satisfied so continuity breaks
- Using this we can prove that there exists a number $\sqrt{2}$ such that $\sqrt{2}\sqrt{2} = 2$, which is much harder to do with CORA alone
  - Take $f(x) = x^2$, which is continuous over $[1, 2]$ by the Polynomial CT
  - $f(1) = 1 < 2 < f(2) = 4$ so by the IVT there exists some number $1 < c < 2$ such that $c^2 = 2$
- We can also define transcendentals: $\sin x = 0$ at $x = \pi$, so take $f(x) = \sin x$ and find a point where $\sin x > 0$ and a point where $\sin x < 0$, so this implies that somewhere in this range there exists a c such that $\sin c = 0$

- The IVT is only true because of the existence of irrational numbers, so it could be used to "create" the irrationals; instead of having CORA, we could have started with an Intermediate Value *Axiom* and then proved the Completeness of the Reals *Theorem*, but this is messy because we would need to first define functions

Bonus: Proof of the IVT

- Proof from pages 50-52 of supplement
- Lemma: Given $f(x)$ continuous on $[a, b]$ and $f(a) < 0 < f(b)$, there exists a number $a < c < b$ for which $f(c) = 0$
  - Let S be the set of all numbers $x \in [a, b]$ such that $f(x) < 0$
  - We know that S contains x in some interval $[a, a + \delta)$
    - Because $\lim\limits_{x \to a^+} f(x)$ exists, if we take $\varepsilon = |f(a)|$, then $|f(x) - f(a)| < |f(a)| = -f(a)$ for $a < x < a + \delta$
    - Therefore $|f(x) - f(a)| < |f(a)| \implies f(a) - |f(a)| < f(x) < f(a) + |f(a)| \implies 2f(a) < f(x) < 0$, and since $f(x) < 0$, $x \in S$
  - This means that S is nonempty and bounded above, so by CORA it has a least upper bound c
  - It is not possible for $f(c) > 0$:
    - Suppose $f(c) > 0$
    - There exists $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < f(c)$ (in this case $\varepsilon = f(c)$, which is assumed positive)
    - $|f(x) - f(c)| < f(c) \implies f(c) - f(c) < f(x) < f(c) + f(c) \implies 0 < f(x)$, when $c - \delta < x \leq c$ or $c \leq x < c + \delta$
    - Since $f(x) > 0$ for $c \in (c - \delta, c)$, these values are not in S and thus are upper bounds of S; but this contradicts the assumption that c is the least upper bound, so $f(c) > 0$ is not possible
  - Similarly $f(c) < 0$ is also not possible, so by process of elimination $f(c) = 0$ (note: $f(c)$ has to exist since $c \in [a, b]$ and $f(x)$ is continuous over that range)
- Now suppose we define $g(x) \equiv f(x) - C$, then $f(a) < C \implies g(a) = f(a) - C < 0$, and $C < f(b) \implies 0 < f(b) - C = g(b)$ and by addition CT $g(x)$ is continuous over $[a, b]$; therefore there exists some c such that $g(c) = 0 \implies f(c) = C$ by the lemma
- Similarly this can be proven for the case where $f(a) > f(b)$

Lecture 9, Sep 29, 2021

The Derivative

- Define the derivative $f'(a) \equiv \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$ if it exists, where $a \in$ domain of $f(x)$
- Example: $f(x) = x^3$, $f'(1)$
  - $$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1 + h)^3 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \\ &= \lim_{h \to 0} \left[h^2 + 3h + 3\right] \\ &= 3 \text{ by the Polynomial LT} \end{aligned}$$
  - h disappears here; it is a dummy variable and the derivative does not depend on it
- With the rigorous definition of the derivative, we can now define rigorously the slope of a tangent, velocity at an instant, etc

Derivative as a Function

- $f'(a)$, the derivative of a, is just a number
- We can define a new function $f'(x) \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$; now both x and h are variables
- Just like before, h will still disappear when the limit is evaluated, so $f'(x)$ is only a function of x
- In evaluating the limit, we treat x as if it were a constant
- If $f'(a)$ exists at a, then we define $f(x)$ to be *differentiable* at a
  - If $f(x)$ is differentiable at all $x \in$ domain of $f(x)$, then we define $f(x)$ to be a *differentiable function*
- Example: Prove $f(x) = x^n \implies f'(x) = nx^{n-1}$ for positive integer n
  - Note the binomial theorem: $$\begin{aligned}&(a+b)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots\end{aligned}$$
  - $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\ &= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \right] \\ &= nx^{n-1} \end{aligned}$$

Lecture 10, Oct 1, 2021

Differentiability

- Definition: $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable for all $x \in (a, b)$
- Definition: $f(x)$ is differentiable on $[a, b]$ if:
  1. $f(x)$ is differentiable on (a, b)
  2. The right hand derivative exists at a
  3. The left hand derivative exists at b
- The left hand derivative is defined as $f'(x^-) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$
- The right hand derivative is the same except the limit is now $h \to 0^+$
- Example: $f(x) = |x|$ is not differentiable at 0, but both the left and right hand derivative exist (but do not equal)
- $f(x)$ may be continuous at a point, but not differentiable; differentiability is "rarer" than continuity
- Differentiability implies continuity:
  - Since continuity guarantees integrability, differentiability guarantees integrability
  - $$\begin{aligned} &\lim_{h\to 0} [f(a+h) - f(a)] \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} h \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \lim_{h\to 0} h \\ &= f'(a) \cdot 0 \\ &= 0 \end{aligned}$$

- $$\begin{aligned} & \lim_{h\to 0}\left[f(a+h)-f(a)\right]=0 \\ \implies & \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) = 0 \\ \implies & \lim_{h\to 0} f(a+h) = f(a) \\ \implies & \lim_{(x-a)\to 0} f(x) = f(a) \text{ substitute } x = a+h \\ \implies & \lim_{x\to a} f(x) = f(a) \\ \implies & f \text{ is continuous at } a \end{aligned}$$
- Note the reverse is not true! Not being differentiable does not imply discontinuity

Vertical Tangent Lines

- Example: $f(x) = \sqrt[3]{x} \implies f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \implies \lim_{x\to 0}|f'(x)| = \infty$
  - Note that $f(x)$ is **not** differentiable at c
- $f(x)$ has a vertical tangent at c if:
  1. $\lim_{x\to c}|f'(x)| = \infty$
  2. $f(x)$ is continuous at c
     - Vertical tangents are different from vertical asymptotes and this separates them
     - e.g. $f(x) = \frac{1}{x^2} \implies f'(x) = -\frac{2}{x^3}$ which is infinite at 0, but this is not a vertical tangent since $f(x)$ is not continuous at this point

Derivative Theorems

- Just like limit theorems, these allow us to compute derivatives without having to evaluate limits
- Some theorems are outlined here:
  1. Constant Derivative Theorem: $f(x) = C \implies f'(x) = 0$
     - Proof is trivial
  2. Additive Derivative Theorem: $(f+g)' = f' + g'$ **if both exist**
     - Proof is trivial
  3. Product Derivative Theorem: $(fg)' = f'g + fg'$
     - $$\begin{aligned} & (fg)' \\ = & \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ = & \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ = & \lim_{h\to 0} \frac{f(x+h)(g(x+h)-g(x)) + g(x)(f(x+h)-f(x))}{h} \\ = & \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h} \\ = & f(x)g'(x) + g(x)f'(x) \end{aligned}$$
  4. Power Derivative Theorem: $f(x) = Cx^n \implies f'(x) = nCx^{n-1}$
  5. Polynomial Derivative Theorem
  6. Reciprocal function Derivative Theorem: $\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}$ for $f(x) \neq 0$

- $$\left(\frac{1}{f(x)}\right)'$$
  $$= \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$
  $$= \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x)f(x+h}$$
  $$= \lim_{h\to 0} \frac{f(x) - f(x+h)}{h} \lim_{h\to 0} \frac{1}{f(x)} \lim_{h\to 0} \frac{1}{f(x+h)}$$
  $$= -f'(x)\frac{1}{f(x)}\frac{1}{f(x)}$$
  $$= -\frac{f'(x)}{f^2(x)}$$

7. Quotient Derivative Theorem: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ for $g(x) \neq 0$
   - Use product and power derivative theorems to prove

- Using ⑥ we can prove ④ works for negative powers:
  - $f(x) \equiv x^{-n}$ where n is a positive integer
  - $g(x) \equiv x^n \implies f(x) = \frac{1}{g(x)}$
  - $f'(x) = -\frac{g'(x)}{g^2(x)} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$

Lecture 11, Oct 4, 2021

Derivatives of Trig Functions

- In cases of $\frac{0}{0}$ our intuition fails
- Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1 - \cos x}{0} = 0$
  - We can't use the product limit theorem because $\lim_{x\to 0} \frac{1}{x}$ does not exist

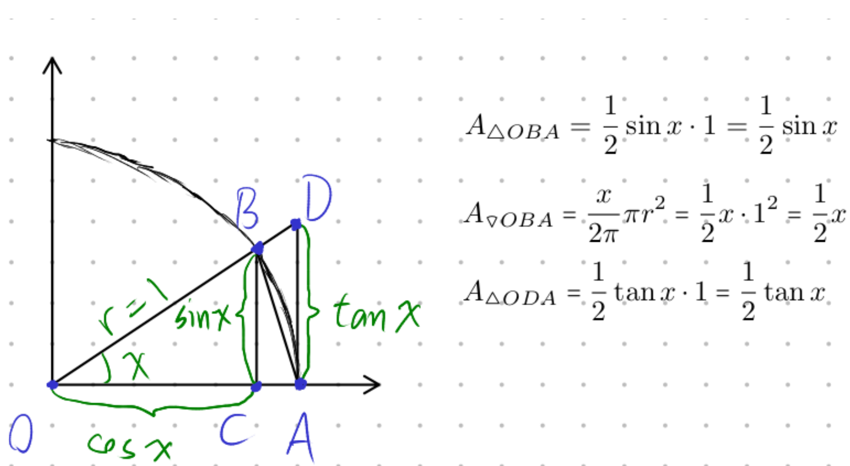


Figure 2: geometric proof

-
- $A_{\triangle OBA} \leq A_{OBA} \leq A_{\triangle ODA} \implies \sin x \leq x \leq \tan x \implies 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \implies \cos x \leq \frac{\sin x}{x} \leq$

1

- We can use the sandwich theorem since $\cos 0 = 1 \implies \lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)}$

$$\begin{aligned}
&= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} \\
&= 1 \cdot \frac{\lim_{x\to 0} -\sin x}{\lim_{x\to 0} \cos x + 1} \\
&= \frac{0}{2} \\
&= 0
\end{aligned}$$

- $$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \sin x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= 0 \cdot \sin x + 1 \cdot \cos x \\
&= \cos x
\end{aligned}$$
- $$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \cos x &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) + \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= 0 \cdot \cos x - 1 \cdot \sin x \\
&= -\sin x
\end{aligned}$$

Chain Rule (Composite Function Derivative Theorem)

- The chain rule theorem: $(f(u(x)))' = f'(u)u'(x)$ or $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$
  - Simple proof, assuming $u(x) \neq u(a)$ when x and a are close:
    * $$\begin{aligned}
(f(u(x)))' &= \lim_{a\to x} \frac{f(u(a)) - f(u(x))}{a - x} \\
&= \lim_{a\to x} \frac{f(u(a)) - f(u(x))}{u(a) - u(x)} \lim_{a\to x} \frac{u(a) - u(x)}{a - x} \\
&= \frac{\mathrm{d}f}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

Implicit Differentiation

- Sometimes there is no explicit expression for $y(x)$ and we only have an implicit relation, e.g. $x^3y^7 - x^2 + y^2 = 0$
- We can find $y'(x)$ here using implicit differentiation and apply the "$\frac{\mathrm{d}}{\mathrm{d}x}$ operator" to both sides
- Using implicit differentiation we can prove that the power derivative theorem works for any rational number
  - Set $u(x) \equiv x^{\frac{p}{q}}$ where p and q are integers; note $u^q = x^p$
  - Set $f(u) \equiv u^q = x^p$; note $f(u(x))$ is a composite function
  - $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \implies px^{p-1} = qu^{q-1}\frac{\mathrm{d}u}{\mathrm{d}x} \implies u' = \frac{px^{p-1}}{qu^{q-1}}$
  - $u = x^{\frac{p}{q}} \implies u^{q-1} = x^{\frac{p}{q}(q-1)} = x^{p-\frac{p}{q}}$
  - $u' = \frac{px^{p-1}}{qu^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{x^{p-\frac{p}{q}}} = \frac{p}{q}x^{\frac{p}{q}-1}$

Higher Derivatives

- Assuming $s'(t)$ exists, it is a function itself, so we can also take its derivative
- $s'(t)$ is the *first derivative*, $s''(t)$ is the *second derivative*, the derivative of the derivative, and so on
- In Newtonian notation $(s'(t))' = s''(t)$, in Leibniz $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ and so on for higher orders
  - With Leibniz we can use unit checking to check for potential mistakes; by putting the exponent on t but not s in $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ we know the units for t are squared but the units for s are not

Lecture 12, Oct 6, 2021

Applications of Derivatives

- Definition: $f(x)$ has an *absolute* maximum at c if $f(c) \geq f(x)$ for all $x \in$ domain $f(x)$
  - **Note $f(x)$ must exist!**
- $f(x)$ has an absolute maximum at c in $[a,b]$ if $f(c) \geq f(x)$ for all $x \in [a,b]$
- $f(x)$ has a local maximum at c if $f(c) \geq f(x)$ for some open interval containing c
  - This interval may be very small but it has to exist on both sides of c

Extreme Value Theorem

- Given $f(x)$ continuous on $[a,b]$, $f(x)$ has an absolute maximum $f(c)$ and an absolute minimum $f(d)$ for some $c, d \in [a,b]$
  - Knowing whether there exists a maximum/minimum is very important because finding maxima/minima is hard!
  - Proof: Supplement pages 52-56
  - Being defined does not imply boundedness; e.g. $\begin{cases} \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is defined everywhere but unbounded
    - Only continuity implies boundedness
  - Part 1: Continuity on an interval implies boundedness
    - Lemma 1: Given $f(x)$ is defined and continuous on $[a,b]$ and $a < c < b$, there exists some interval $(c-\delta, c+\delta)$ with $\delta > 0$, in $[a,b]$ for which $f(x)$ is bounded
      - This can be easily proven using continuity and epsilon delta; taking $\varepsilon = 1$ implies that on some interval $c - \delta < x < c + \delta$, $|f(x) - f(c)| < 1$
    - Lemma 2: Given $f(x)$ is defined and continuous on $[a,b]$, $f(x)$ is bounded on $[a,b]$
      - Let S be the set of all u for which $f(x)$ is bounded on $[a,u]$
      - Using Lemma 1, $f(x)$ is bounded on some interval $[a,u]$ with $u > a$, so S is nonempty; as $u < b$, S is bounded above by b and so by CORA has a least upper bound c

- Suppose $c < b$, then by Lemma 1 there exists $\delta > 0$ such that $f(x)$ is bounded on $(c-\delta, c+\delta)$; therefore, $f(x)$ is bounded on $[a, c+\delta)$, which contradicts the statement that c is the least upper bound of S (since if this were true, the least upper bound of S would be $c+\delta$); therefore $c = b$
- Now we know that $f(x)$ is bounded on $[a, b)$ (b is not closed because the least upper bound is not necessarily a member of the set)
- Since f is continuous at b, it is bounded at b; then by adapting Lemma 1 to an endpoint we know f is bounded on $(b-\delta, b]$; since $f(x)$ is bounded on both $[a, b)$ and $(b-\delta, b]$, $f(x)$ is bounded on $[a, b]$

- Part 2: Continuity on an interval implies the existence of a maximum and minimum
  * Consider the set $S = \{f(x) : a \leq x \leq b\}$; then S is bounded above since $f(x)$ is bounded and therefore by CORA it has a least upper bound M, so $f(x) \leq M$ for all $x \in [a, b]$
  * Now we need to prove that $f(c) = M$ since a maximum requires that the function takes on that value
  * Suppose $f(x)$ never equals M, then we can define $g(x) \equiv \dfrac{1}{M - f(x)}$, so $g(x)$ is always positive and defined on $[a, b]$, so $g(x)$ is continuous by the additivity and quotient continuity theorems
    - Therefore, $g(x)$ is also bounded on $[a, b]$ by part 1, so there exists some number K such that $0 < g(x) \leq K$, and $K > 0$
    - $\dfrac{1}{K} \leq \dfrac{1}{g(x)} = M - f(x) \implies f(x) \leq M - \dfrac{1}{K}$, but this violates the fact that M is the least upper bound of S since $M - \dfrac{1}{K}$ is smaller
  * Therefore, by contradiction, there is some $c \in [a, b]$ such that $f(c) = M$

- Note that being bounded is not the same as having an absolute maximum/minimum; e.g. $\dfrac{\sin x}{x}$ is bounded but does not have a maximum
  - $\dfrac{\sin x}{x}$ is bounded by 1, but we cannot say that the maximum is 1, because it is undefined at 0 and so never takes on the value of 1

Fermat's Theorem

- Definition: c is a critical point of $f(x)$ if $f'(c) = 0$ or DNE
- Fermat's Theorem: Given $f(x)$ has a local maximum or minimum at c, then c is a critical point
  - The reverse is not true! $f'(c) = 0$ or DNE does not always imply that c is a local minimum or maximum
  - Note: This does not apply for maximum or minimum at end points of a range
  - Proof: textbook page 212
    * Suppose $f(c)$ is a local maximum; then $f(c) \geq f(x)$ or $f(c) \geq f(c+h)$ for some small h, positive or negative, so $f(c+h) - f(c) \leq 0$
    * Suppose h is positive, then:
      - $\dfrac{f(c+h) - f(c)}{h} \leq 0 \implies \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} \leq 0 \implies f'(c) \leq 0$
    * Suppose h is negative, then:
      - $\dfrac{f(c+h) - f(c)}{h} \geq 0 \implies \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} \geq 0 \implies f'(c) \geq 0$
      - Note here the direction of the inequality is flipped since we divided by h, a negative quantity
    * Since both $f'(c) \geq 0$ and $f'(c) \leq 0$, $f'(c) = 0$ must be true if it exists; alternatively it may be undefined
- Given a continuous $f(x)$ on $[a, b]$, by the extreme value theorem an absolute maximum/minimum of the range must exist; to test for the absolute maximum/minimum:
  1. Find all critical points c and $f(c)$
  2. Find the endpoints $f(a)$, $f(b)$
  3. The largest f value is the absolute maximum, the minimum f value is the absolute minimum

Lecture 13, Oct 8, 2021

The Mean Value Theorem

- *Rolle's Theorem*: Given $f(x)$ continuous on $[a, b]$, differentiable on $[a, b)$, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$ (there may be more than one)
  1. $f(x) > f(a) = f(b)$ for some $x \in (a, b)$ (i.e. the value at x is greater than the value at the end points a and b)
     - By the Extreme Value Theorem there exists an absolute maximum $f(c)$ somewhere in $[a, b]$, which cannot be an endpoint max (because $f(x) > f(a)$)
     - Therefore the maximum must be a local maximum
     - Then by Fermat's theorem c is a critical point
     - $f(c)$ cannot be undefined because $f(x)$ is differentiable on $[a, b)$ so the derivative exists for all values in $[a, b)$
     - Therefore $f'(c) = 0$
  2. $f(x) < f(a) = f(b)$ for some $x \in (a, b)$
     - Similar to (1), there exists a minimum which is not an endpoint minimum so it is a local minimum; therefore it is a critical point; the derivative at this minimum cannot be undefined, therefore the value of f' at this critical point must be 0
  3. $f(x) = f(a) = f(b)$ (constant)
     - Since $f(x)$ is a constant then $f'(x) = 0$ by the constant derivative theorem
- The *Mean Value Theorem* is a generalization of Rolle's theorem: Given $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, i.e. the derivative at c equals the mean rate of change of f over $[a, b]$
  - Define $y(x)$ to be the secant line from $(a, f(a))$ to $(b, f(b))$ and $g(x) \equiv f(x) - y(x)$
  - Then $g(x)$ is continuous over $[a, b]$ because both $f(x)$ is continuous (as given) and $y(x)$ is continuous (by the polynomial continuity theorem)
  - $g(x)$ is differentiable over (a, b) because both $f(x)$ is differentiable (as given) and $y(x)$ is differentiable (by the polynomial derivative theorem)
  - $g(a) = g(b) = 0$ since at a and b the secant line intersects $f(x)$
  - Therefore $g(x)$ meets the requirements for Rolle's Theorem and so there exists some $c \in (a, b)$ such that $g'(c) = 0$
  - $g'(c) = 0 \implies f'(c) - y'(c) = 0 \implies f'(c) = y'(c) = \dfrac{f(b) - f(a)}{b - a}$

The Structure of Calculus Logic

- Define numbers with the field and order axioms →rationals →functions →limits →
  - Continuity
  - Derivatives
- CORA →irrationals $\xrightarrow{\text{+continuity}}$ IVT, EVT $\xrightarrow{\text{+derivatives}}$ MVT

Differentials

- Suppose $y = f(x)$; define two new variables Δy and Δx related by $\Delta y \equiv f(x + \Delta x) - f(x)$
  - Note the value of Δy depends on **both** Δx and x
- Define *differentials* $\mathrm{d}x$ and $\mathrm{d}y$ related by $\mathrm{d}y \equiv f'(x)\,\mathrm{d}x$
- Note that since $\mathrm{d}x$ and Δx are both free variables, we can set $\mathrm{d}x = \Delta x$
  - Then Δy is the true value and $\mathrm{d}y$ is a tangent-line approximation of f which improves as $\Delta x \to 0$
  - Practical note: Sometimes we want Δy but it might not be straightforward to compute it, so we can use $\mathrm{d}y$ as an approximation if Δx is small
- Note: The $\mathrm{d}x$ and $\mathrm{d}y$ here are not the same as the ones in $\dfrac{\mathrm{d}y}{\mathrm{d}x}$; the derivative is not a fraction, but the $\mathrm{d}y$ and $\mathrm{d}x$ here *are* numbers; it's just bad notation

Lecture 14, Oct 13, 2021

Bounding Estimations Using the Mean Value Theorem

- From last lecture we used differentials to estimate values: $29^{\frac{1}{3}} \approx 3.074$ using $f(x) = \sqrt[3]{x}$ and $x = 27$, $\Delta x = 2$
- Now we can use the MVT to bracket our estimate
- If we apply MVT to $f(x) = \sqrt[3]{x}$ on $[27, 29]$:
  - There is some $c \in (27, 29)$ such that $f'(c) = \dfrac{\sqrt[3]{29} - 3}{2}$
  - Since $f'(c) = \dfrac{1}{3}c^{-\frac{2}{3}}$
  - Therefore $\sqrt[3]{29} = \dfrac{2}{3}c^{-\frac{2}{3}} + 3$
  - Since we know $27 < c < 29$, we now have bounds on $\sqrt[3]{29}$; when $c = 27$, $\dfrac{2}{3}c^{-\frac{2}{3}} + 3 = \dfrac{2}{3} \cdot \dfrac{1}{9} + 3 = \dfrac{2}{27} + 3$ so c cannot exceed that
  - When $c = 29$ we can estimate $29^{-\frac{2}{3}}$, since $29 < 64 \implies 29^{\frac{2}{3}} < 64^{\frac{2}{3}} = 16 \implies 29^{-\frac{2}{3}} > \dfrac{1}{16}$ so $\sqrt[3]{29} < \dfrac{2}{3} \cdot \dfrac{1}{16} + 3$
  - Therefore $3.0416 < \sqrt[3]{29} < 3.074$

Derivatives and Graphing: 4 Quick Tests

1. Increasing/decreasing test
   - Given f is differentiable on I, if $f' > 0$ over I, f is increasing; if $f' < 0$, f is decreasing; else if $f' = 0$, f is constant
   - Proof: Using the MVT
     - Consider any $x_1 < x_2 \in I$, since f is differentiable, the MVT holds
     - Therefore there is some $c \in I$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, but as f' is positive over I, $f(x_2) > f(x_1)$, so by definition f is increasing
     - Proof is similar for the two other cases
2. First Derivative Test
   - Since $f(c_{crit})$ includes maxima, minima, and other values, we want to be able to know what values to keep
   - Given I contains a critical point c, f continuous at c, and f differentiable in I but not necessarily at c:
     1. If $f' > 0$ just to the left of c and $f' < 0$ just to the right of c, then c is a local maximum
     2. If $f' < 0$ just to the left of c and $f' > 0$ just to the right of c, then c is a local minimum
     3. If f' does not change sign, then c is neither a maximum nor a minimum
   - Proof of (1):
     - The statement of (1) means that there is some a such that $f' > 0$ for $x \in (a, c)$, so by quick test 1 f is increasing on (a, c)
     - Therefore $f(c) \geq f(x)$ for all $x \in (a, c)$
     - By the same logic there is some b such that $f(c) \geq f(b)$ for all $x \in (c, b)$
     - Therefore $f(c) \geq f(x)$ for $x \in (a, b)$, which by definition is a local maximum
   - (2) and (3) could be proven similarly
   - Note that continuity is strictly required
3. Concavity Test (for definition refer to next section)
   - Given $f(x)$ is twice differentiable on I, $f'' > 0 \implies f$ is concave up; $f'' < 0 \implies f$ is concave down
   - Proof: Textbook A46
     - Suppose a is the point of interest and $f''(a) > 0$; we need to show that the function lies above the tangent, i.e. $f(x) > f(a) + f'(a)(x - a)$ for $x \in I$ and $x \neq a$
     - Suppose $x > a$:

        * Applying the MVT we have $f(x) - f(a) = f'(c)(x-a)$ where $c \in (a, x)$
        * Since $f'' > 0$ we know f' is increasing on this interval, so $f'(a) < f'(c) \implies f'(a)(x-a) < f'(c)(x-a) \implies f(a) + f'(a)(x-a) < f(a) + f'(c)(x-a)$
        * But $f(x) = f(a) + f'(c)(x-a)$ because of the MVT, so $f(a) + f'(a)(x-a) < f(x)$, therefore the function lies above the tangent and it is concave up
        * Other cases can be proved similarly
4. Second Derivative Test
    - Given f'' exists and is continuous near c, and $f'(c) = 0$ then if $f''(c) > 0$ then $f(c)$ is a local minimum; if $f''(c) < 0$ then $f(c)$ is a local maximum; if $f''(c) = 0$ this test is inconclusive

Concavity and Points of Inflection

- Definition: If the graph of $y = f(x)$ lies above all its tangents in I, then $f(x)$ is concave up in I; similarly if it lies below all its tangents then $f(x)$ is concave down
- Definition: $f(x)$ has a point of inflection at c if $f(x)$ is continuous at c and the sign of concavity (up/down) changes at c

Lecture 15, Oct 15, 2021

Horizontal Asymptotes/Limits at Infinity

- Definition: $\lim_{x \to \infty} f(x) = L$ means that $\forall \varepsilon > 0, \exists A$ such that $x > A \implies |f(x) - L| < \varepsilon$; this is a *horizontal asymptote* of $f(x)$
    - i.e. we can find f values as close to L as possible by going to large enough x values
- Limit theorems also exists for limits at infinity just like regular limits
- Theorem: $\lim_{x \to \infty} \frac{1}{x^r} = 0$ for $r > 0$
    - Proof: For all $\varepsilon > 0$, choose $A = \varepsilon^{-\frac{1}{r}}$
        * $$\begin{aligned} & x > A = \varepsilon^{-\frac{1}{r}} \\ \implies & x^r > \varepsilon^{-1} \\ \implies & \frac{1}{x^r} < \varepsilon \\ \implies & \left|\frac{1}{x^r}\right| < \varepsilon \end{aligned}$$
        * Thus $x > A \implies \left|\frac{1}{x^r} - 0\right| < \varepsilon$, so $\lim_{x \to \infty} \frac{1}{x^r} = 0$

Slant Asymptotes

- Definition: If $\lim_{x \to \infty} [f(x) - (mx + b)] = 0$, then the line $y = mx = b$ is a slant asymptote to $f(x)$ at $+\infty$
    - As always this does not mean that the limit at infinity exists, but it gives us more information than simply saying that $\lim_{x \to \infty} f(x)$ DNE

Max/Min Problems

- General checklist for max/min problems:
    1. Critical points
        - Find derivative, second derivative, determine concavity if it helps
    2. Endpoints
        - Find the domain, as constrained by the question statement
        - Find value at end points
    3. Check interior points
        - Find value at internal local extrema

4. Check limit at $\pm\infty$ if the question allows it
5. Compare values and decide the max/min
   - Make sure to state the answer!

- Example: A string 28cm long is to be cut into 2 parts to make a square and a circle. Where should the cut be made to get a) the maximum area and b) the minimum area?
  - Let the square have side x and the circle have radius y
  - We know that $28 = 4x + 2\pi y$ therefore we can relate x and y by $y = 7 - \frac{\pi}{2}x$, which allows us to eliminate one of the variables
  - The total area is then $A(x, y) = x^2 + \pi y^2 \implies A(y) = \left(7 - \frac{\pi}{2}y\right)^2 + \pi y^2$
  - At this point we need to **establish the domain** since the max/min can be at an endpoint
    * $y \geq 0$ and $x \geq 0$ due to physical constraints, so the domain is $\left[0, \frac{14}{\pi}\right]$
  - $A(0) = 49,\ A\left(\frac{14}{\pi}\right) = \pi\frac{14^2}{\pi^2} \approx 60$
  - Note A is a polynomial of y and therefore is continuous by the polynomial continuity theorem; therefore by the EVT for this *closed* interval there is an absolute max/min
    * Sometimes there is not an absolute max/min!
    * Note the importance of establishing the domain right away, so we know the function is continuous over the closed interval
  - $A'(y) = -\pi\left(7 - \frac{pi}{2}y\right) + 2\pi y = -7\pi + y\left(\frac{\pi^2}{2} + 2\pi\right)$
  - $A''(y) = \frac{\pi^2}{2} + 2\pi > 0$ therefore $A(y)$ is concave up
  - $A'(y) = 0 \implies y_{crit} = \frac{7}{2 + \frac{\pi}{2}} \approx 2$
  - $x_{crit} \approx 7 - \pi \approx 4$
  - $A(y_{crit}) \approx 28$
  - Since $A\left(\frac{14}{\pi}\right) > A(0) > A(y_{crit})$, the absolute max is $A\left(\frac{14}{\pi}\right)$ and the absolute min is $A\left(\frac{14}{\pi}\right)$
    * i.e. the max occurs when not cutting the string and using it all for the circle; the min occurs when cutting the string at $\frac{14}{\pi}$ (about 12 and 16cm lengths) for the square and circle respectively
  - Since A' is a polynomial it always exists

Lecture 16, Oct 18, 2021

Curve Sketching

- Checklist:
  1. Find:
     - Domain, range
     - $\lim\limits_{x\to\pm\infty} f(x)$
     - End points if they exist
     - Vertical/horizontal/slant asymptotes
  2. Find intercepts (x value where $f = 0$, f value where $x = 0$)
  3. Establish whether $f(x)$ is:
     - Even or odd
     - Periodic
  4. Find $f'(x)$, then:
     - Find all critical points c and $f(c)$ and find local max/min
     - Find ranges where $f(x)$ is increasing and decreasing
     - Find vertical tangents (reminder: vertical tangent is when derivative is infinity but function is continuous)

5. Find $f''(x)$, then:
   - Find where $f(x)$ is concave up/down and points of inflection
     * Note: $f(x)$ has to be continuous at a point of inflection
   - Use the second derivative test to confirm local max/min from the last step
6. Find the absolute maximum/minimum if they exist

- Example: $f(x) = 2x^3 - 3x^2$ (note $f(x) = x^2(2x-3)$)
  1. Domain is all x, no asymptotes, no end points, $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = -\infty$ so range is all reals
  2. $f(x) = 0$ when $x = 0, \frac{3}{2}$, $x = 0$ when $f = 0$
  3. Not symmetric or periodic
  4. $f'(x) = 6x^2 - 6x = 6x(x-1)$
     - Critical points: $x = 0, 1$
     - $f' > 0$ for $x > 1 \implies f(x)$ increasing for $x > 1$
     - $f' < 0$ for $0 < x < 1 \implies f(x)$ decreasing for $0 < x < 1$
     - $f' > 0$ for $x < 0 \implies f(x)$ increasing for $x < 0$
     - $f(1)$ is a local minimum, $f(0)$ is a local max
     - No vertical tangents or cusps
  5. $f''(x) = 12x - 6 = 6(2x-1)$
     - Inflection point at $x = \frac{1}{2}$
     - $f'' > 0$ for $x > \frac{1}{2}$ so f is concave up for $x > \frac{1}{2}$
     - $f'' < 0$ for $x < \frac{1}{2}$ so f is concave down for $x < \frac{1}{2}$
  6. No absolute max/min since range is all reals

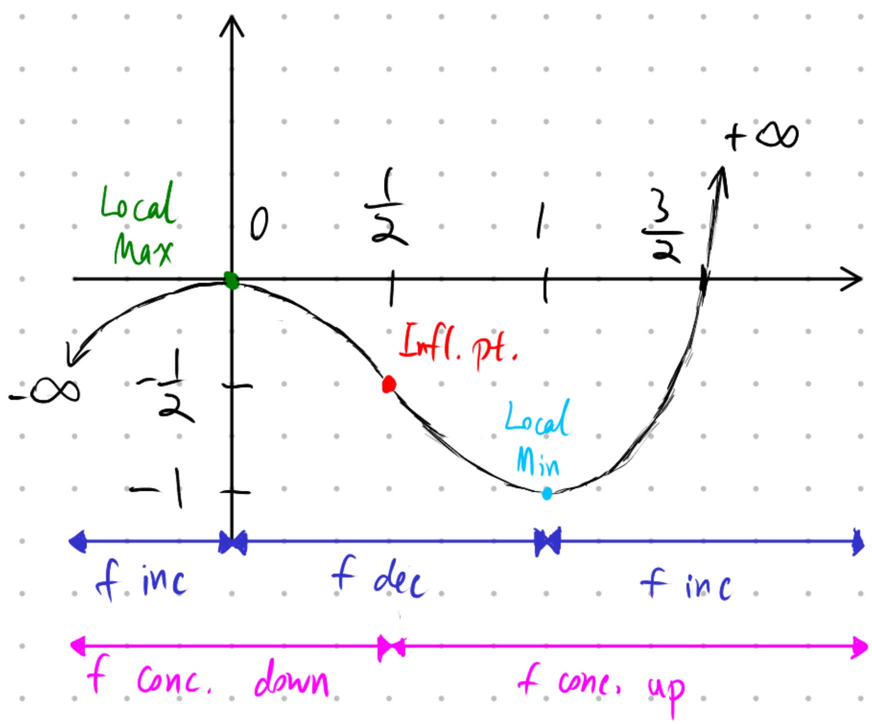


Figure 3: Graph of $f(x) = 2x^3 - 3x^2$

Numerical Root Finding Methods

- Sometimes we might want to find the roots of a complicated polynomial; for polynomials of degree 5 and above no formula exists, but there exist numerical methods that can get us approximate values
- Method of Successive Bisections
  - By trial and error, find $a < b$ such that $f(a) > 0 > f(b)$ or $f(a) < 0 < f(b)$
  - If f is continuous, then by the IVT, $f(c) = 0$ for some $a < c < b$
  - For each step:
    1. Calculate the halfway point between $a_i + b_i$, denoted x_{hi}
    2. Calculate $f(x_{hi})$; if $f(x_{hi}) > 0$ then we can pick $a_{i+1} = x_{hi}$ and keep b the same
       * Conversely if $f(x_{hi}) < 0$ we keep a the same but pick $b_{i+1} = x_{hi}$
  - For each step the possible range for c is cut in half; eventually the range is small enough that the difference between a and b does not matter
  - This method always converges
- Newton's Method
  - Easier to use and faster
  - However, $f(x)$ must be differentiable, and this method may not converge if initial guess is too far off
  - Procedure:
    1. Start with an initial guess of x_1
       * This must be a good guess, otherwise will not converge
    2. Compute $f(x_1)$ and $f'(x_1)$ to find the tangent line: $y_t(x) = f(x_1) + f'(x_1)(x - x_1)$
    3. Find where the tangent line approximation intersects the x-axis: $f(x_1) + f'(x_1)(x_2 - x_1) = 0 \implies x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$
    4. Repeat the procedure until estimate is as close to the true root as desired
  - If the tangent line at x is not a good approximation of the function's behaviour, then the next iteration can give a worse number (e.g. if there is a local max between your guess and the root)
  - This method may also fail for some particular functions, e.g. $x^{\frac{1}{3}}$
- Generally, try Newton's Method first, and if it does not converge, try different guesses or try the bisection method

Antiderivatives

- Definition: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$
- Unlike derivatives, not every function has an antiderivative; (educated) guessing is always involved
- The test for whether $F(x)$ is an antiderivative of $f(x)$ is the rigorous part, but finding $F(x)$ itself is not (like limits)
- $F(x) + C$ is the most general antiderivative of $f(x)$ because antiderivatives are not unique
- Properties to be memorized:
  1. $af(x) \to aF(x) + C$
  2. $f(x) + g(x) \to F(x) + G(x) + C$
  3. $x^n \to \dfrac{1}{n+1}x^{n+1} + C$ for $n \neq -1$
  4. $\cos x \to \sin x + C$
  5. $\sin x \to -\cos x + C$
  6. $\sec^2 x \to \tan x + C$

Lecture 17, Oct 20, 2021

Summation Theorems

1. Constant multiple: $\sum \alpha a_i = \alpha \sum a_i$
2. Additivity/distributive: $\sum (a_i + b_i) = \sum a_i + \sum b_i$

3. $\sum_{i=1}^{n} 1 = n$
4. $\sum_{i=1}^{n} \alpha = \alpha n$
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
8. $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Limits of Sums

- If $f(n) = \sum_{i=1}^{n} a_i$, then we can take $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$
- Example:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{5}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{5}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{5}{6} \lim_{n\to\infty} \frac{2n^3+3n^2+n}{n^3} \\
&= \frac{5}{6} \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&= \frac{5}{3}
\end{aligned}$$

Area Under a Curve

- How do we define the area under a curve?
- Definition: A *partition* is a way to divide a closed interval $[a, b]$ into a finite number of subsets, including a and b
  - e.g. $[x_0, x_1], [x_1, x_2], \cdots$
  - We can then define $\Delta x_i \equiv x_i - x_{i-1}$
- Definition: The *norm* of a partition P is $\|P\|$, defined as the length of the longest subinterval
- In every subinterval there is an x_i^* which lies within the interval, which makes the area of each rectangle $A_i = f(x_i^*)\Delta x_i$
- The total area is $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$
  - We can't simply make $n \to \infty$ because the partitions may not be equal in size
- In practice we usually work with equal sized intervals, but sometimes in numerical integration it might be useful to vary the size of the subintervals
- Example: $y = \cos x$, $0 \le x \le b \le \frac{\pi}{2}$ with a regular partition $\Delta x_i = \frac{b}{n} = \|P\|$

Lecture 18, Oct 22, 2021

Defining the Definite Integral

- Definition: If f is a function defined on $[a,b]$, let P be a partition of $[a,b]$ with partition points $a = x_0 < x_1 < \cdots < x_n = b$, choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$, and $\|P\| = \max\{\Delta x_i\}$, then the *definite integral* of f from a to b is $\displaystyle\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ if the limit exists
  - This is called the *Riemann* definition of an integral, since $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is a *Riemann sum*
  - More precisely $\displaystyle\int_a^b f(x)\,\mathrm{d}x = I \iff \forall\varepsilon > 0, \exists\delta > 0$ such that $\|P\| < \delta \implies \left|I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i\right| < \varepsilon$ for all partitions P of $[a,b]$ and all possible choices of $x_i^* \in [x_{i-1}, x_i]$
  - An integral doesn't need to represent an area; it can also represent other things and is a very general process
  - If the integral does represent an area, then it is either the area under a curve if the function is always positive, or a difference of areas if the function becomes negative at some point
- If $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ exists, then f is *integrable* over the interval $[a,b]$
- Even though the limit might not be evaluable analytically, the definite integral can still be approximated to any degree of accuracy desired
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$ and $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
- Theorem: Piecewise continuity implies integrability
  - Piecewise continuity means that there is only a finite number of jump discontinuities (this does not include infinite discontinuities)
  - Proof requires more background in series and sequences so comes later
  - If f goes to infinity at some point in the interval then the integral may or may not exist
- Practically, for continuous functions, we assume:
  1. Regular partition $\Delta x = \Delta x_i = \dfrac{b-a}{n}$
  2. Right hand or left hand end point: $x_i^* = x_i = a + i\Delta x = a + i\dfrac{b-a}{n}$
- With these assumptions, $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}$

Integral Properties

- Just like limits, we have theorems for integrals; in fact, because the definite integral is defined as a limit, many of these are just limit properties
- Properties of integrals:
  1. Constant: $\displaystyle\int_a^b c\,\mathrm{d}x = c(b-a)$
  2. Sum: $\displaystyle\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
  3. Constant multiple: $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$
  4. Combining bounds: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^z f(x)\,\mathrm{d}x + \int_z^b f(x)\,\mathrm{d}x$, note that z does not have to be between a and b!
- Order properties:
  1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$
4. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$

Lecture 19, Oct 25, 2021

The Fundamental Theorem of Calculus

- Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$
  - Example: $f(x) = x \implies \int_0^x t\,\mathrm{d}t = \frac{1}{2}x^2 = F(x)$
  - Notice that $F'(x) = x = f(x)$; is this true in general?
- F is the area under $f(x)$ from a to x, so for small h, $F(x+h) - F(x)$ is approximately the area of the small rectangle of width h and height $f(x)$, so intuitively $\frac{F(x+h) - F(x)}{h} \approx f(x)$
- Theorem: Let f be a continuous function on $[a, b]$, defined $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then F is continuous on $[a, b]$, differentiable on (a, b) and has derivative $F'(x) = f(x)$ for all $x \in (a, b)$
  - Proof: For $x, x+h \in (a, b)$, $F(x+h) - F(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t = \int_x^{x+h} f(t)\,\mathrm{d}t$
  - Then $\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t$
  - Consider $h > 0$:
    * By the EVT, there exists minimum $f(u) = m$ and maximum $f(v) = M$ for $u, v \in [x, x+h]$; then by the order theorems,
    $$\begin{aligned} f(u)h &\leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq f(v)h \\ \implies f(u) &\leq \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \leq f(v) \\ \implies f(u) &\leq \frac{F(x+h) - F(x)}{h} \leq f(v) \end{aligned}$$
    * As $x \leq u, v \leq x+h$, $\lim_{h \to 0} u = \lim_{h \to 0} v = x \implies \lim_{h \to 0} f(u) = \lim_{h \to 0} f(v) = f(x)$
    * Thus by the squeeze theorem $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$, i.e. $F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$
- The fundamental theorem of calculus: Let f be continuous on $[a, b]$, then if G is any antiderivative of f on $[a, b]$ then $\int_a^b f(t)\,\mathrm{d}t = G(b) - G(a)$
  - Proof: $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f by the previous theorem; then $F'(x) = G'(x) \implies F(x) = G(x) + C$
  - $F(a) = \int_a^a f(t)\,\mathrm{d}t = 0 \implies G(a) + C = 0 \implies C = -G(a) \implies F(x) = G(x) - G(a)$
  - Then $\int_a^b f(t)\,\mathrm{d}t = F(b) = G(b) - G(a)$

Net Change and Indefinite Integrals

- $\int_a^b F'(x)\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x \approx \sum \mathrm{d}F = \Delta F = F(b) - F(a)$

- By taking an integral, we are summing up very small pieces of change to get the net change
- Example: Integrating the velocity $v(t)$ to get a change in position
- We can leave out the bounds and get $\int f(x)\,\mathrm{d}x = F(x) + C$, which is an *indefinite integral*
- An indefinite integral is a family of curves since the value of C can vary
- Example: $$\begin{aligned}\int (2+\tan^2\theta)\,\mathrm{d}\theta &= \int (1+1+\tan^2\theta)\,\mathrm{d}\theta \\ &= \int (1+\sec^2\theta)\,\mathrm{d}\theta \\ &= \theta + \tan\theta + C\end{aligned}$$
- If we are given an initial condition, we can use that to determine a value for C
- Example: Suppose that a particle has an acceleration of $a(t) = 3 - t$, and $x(0) = 2, x(3) = -1$, where is the particle at $t = 6$? How far has the particle travelled in the 6 seconds?
  - $v(t) = \int a(t)\,\mathrm{d}t = 3t - \frac{1}{2}t^2 + C$
  - $x(t) = \int v(t)\,\mathrm{d}t = \frac{3}{2}t^2 - \frac{1}{6}t^3 + Ct + k$
  - From the known values of $x(t)$, we can find C and k
  - $x(0) = k = 2 \implies k = 2$
  - $x(3) = \frac{3}{2}9 - \frac{1}{6}27 + 3C + 2 = -1 \implies C = -4$
  - Therefore $x(t) = \frac{3}{2}t^2 - \frac{1}{6}t^3 - 4t + 2$, thus at $t = 6$, the particle is at $x(t) = -4$
  - To find the distance travelled, we need to integrate the absolute velocity since the particle can have a negative velocity between $t = 0$ to 6
  - $s = \int_0^6 |v(t)|\,\mathrm{d}t = \int_0^6 \left|-\frac{1}{2}(t-2)(t-4)\right|\mathrm{d}t$
  - The particle changes direction twice; for $t \in (0,2)$, $v(t) < 0$; for $t \in (2,4)$, $v(t) > 0$; for $t > 4$, $v(t) < 0$, so we can use this to break up the integral
  - $$\begin{aligned}\int_0^6 |v(t)|\,\mathrm{d}t &= \int_0^2 -v(t)\,\mathrm{d}t + \int_2^4 v(t)\,\mathrm{d}t + \int_4^6 -v(t)\,\mathrm{d}t \\ &= [-x(t)]_0^2 + [x(t)]_2^4 - [x(t)]_4^6 \\ &= x(0) - x(2) + x(4) - x(2) + x(4) - x(6) \\ &= 7 + \frac{1}{3}\end{aligned}$$

Lecture 20, Oct 27, 2021

The Substitution Rule

- We can use the chain rule in reverse, known as u-substitution
- Since $\frac{\mathrm{d}}{\mathrm{d}x}f(g(x))) = f'(g(x))g'(x)$, we can reverse this
- If we have $\int f(g(x))g'(x)\,\mathrm{d}x$, make the substitution that $\mathrm{d}u = g'(x)\,\mathrm{d}x$, then $$\begin{aligned}\int f(g(x))g'(x)\,\mathrm{d}x &= \int f(u)\,\mathrm{d}u \\ &= F(u) + C \\ &= F(g(x)) + C\end{aligned}$$
- Example: $\int \frac{x}{\sqrt{x^2-4}}\,\mathrm{d}x$, let $u = x^2 - u$ and $\mathrm{d}u = 2x\,\mathrm{d}x$, then $$\begin{aligned}\int \frac{x}{\sqrt{x^2-4}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= \frac{1}{2}\cdot 2u^{\frac{1}{2}} + C \\ &= \sqrt{x^2-4} + C\end{aligned}$$
- Recognizing the appropriate substitution to make requires experience

- For definite integrals, we can eliminate the back substitution step by using a *change of variables formula*: $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$
  - Proof: Given $F' = f$, $$\begin{aligned}&\int_a^b f(g(x))g'(x)\,\mathrm{d}x \\ &= \int_a^b F'(g(x))g'(x)\,\mathrm{d}x \\ &= [F(g(x))]_a^b \\ &= F(g(b)) - F(g(a)) \\ &= [F(u)]_{g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u\end{aligned}$$
  - Example: $\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x$, let $u = x^2+1, \mathrm{d}u = 2x\,\mathrm{d}x \implies u(0) = 1, u(1) = 2$, so $$\begin{aligned}\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x &= \int_1^2 \frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u \\ &= \left[\frac{u^{\frac{3}{2}}}{3}\right]_1^2 \\ &= \frac{2\sqrt{2}}{3} - \frac{1}{3}\end{aligned}$$
- Substitution can be used even in instances that aren't exactly $f(g(x))g'(x)$ to simplify things
  - Example: $\int x^2(2x+1)^2\,\mathrm{d}x$, we can substitute $u = 2x+1, \mathrm{d}u = 2\,\mathrm{d}x$, and $x = \frac{1}{2}(u-1)$, so the integral becomes $\frac{1}{2}\int\left(\frac{1}{2}(u-1)\right)^2 u^4\,\mathrm{d}u$. This still needs to be expanded out, but now we only have to expand the quadratic instead of the quartic

Using Symmetry to Evaluate Integrals

1. If f is odd on $[-a, a]$ then $\int_{-a}^a f(x)\,\mathrm{d}x = 0$
2. If f is even on $[-a, a]$ then $\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^2 f(x)\,\mathrm{d}x$

Lecture 21, Oct 29, 2021

Area Between Curves

- Suppose $f(x) \geq g(x)$ and continuous on $x \in [a, b]$, then the area between them can be partitioned just like the area under a single curve into $A \approx \sum_{i=1}^n [f(x_i^*) - g(x_i^*)]\,\Delta x_i$; taking the limit, $A = \lim_{\|P\|\to 0} \sum_{i=0}^n [f(x_i^*) - g(x_i^*)]\,\Delta x_i = \int_a^b [f(x) - g(x)]\,\mathrm{d}x$
- For positive functions this can be interpreted as the difference between the areas under two curves
- Example: Find the area between $y = x+2$ and $y = 4-x^2$
  - Find the intersection: $4 - x^2 = x + 2 \implies x^2 + x - 2 = 0 \implies (x-1)(x+2) = 0$, so the intersections are at $(1, 3)$ and $(-2, 0)$
  - On this interval $x + 2 > 4 - x^2$

- $$\int_{-2}^{1}(4-x^2-(x+2))\,\mathrm{d}x = \int_{-2}^{1}(-x^2-x+2)\,\mathrm{d}x = \left[-\frac{x^3}{3}-\frac{x^2}{2}+2x\right]_{-2}^{1} = \frac{9}{2}$$
- Need to be careful when curves cross since the curve that's the upper bound may change; generally the area is $A = \int_a^b |f(x)-g(x)|\,\mathrm{d}x$
- The independent variable does not have to be x

Volumes

- To find volumes, we can break up the region into very small slices and add up all the little bits of area
- $V_i \approx A_i\Delta x_i \implies V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$, and in the limit $V = \int_a^b A(x)\,\mathrm{d}x$
- Example: Rectangular pyramid on the x axis, h units tall, with a base width of r
  - The radius at each x is $\frac{r}{2h}x$, since if you look at it from the side, the edge of the pyramid is $\frac{r}{2}$ units above the x axis and h units from the y axis, so the slope is $\frac{r}{2h}$
  - Therefore $A(x) = \left(\frac{rx}{h}\right)^2 = \frac{r^2x^2}{h^2}$
  - $$V = \int_0^h \frac{r^2x^2}{h^2}\,\mathrm{d}x = \frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{1}{3}r^2h$$
- Example: Calculate the volume of a sphere of radius r, at the origin ($x^2+y^2=r^2$)
  - If we take a slice, the radius here would be $y = \sqrt{r^2-x^2}$, so the area of the slice would be $A = \pi(r^2-x^2)$
  - $$V = \int_{-r}^{r} \pi(r^2-x^2)\,\mathrm{d}x = \pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}$$

Lecture 22, Nov 1, 2021

Solids of Revolution

- A solid of revolution is a solid obtained by revolving a region about an axis, usually (but doesn't have to be) the x axis
- The area of each slice is then $A_i = \pi(f(x))^2$ so $V = \int_a^b \pi(f(x))^2\,\mathrm{d}x$, known as the disk method
  - Example: Volume of cone with height h and radius r
    - $y = \frac{r}{h}x \implies V = \int_0^h \pi\left(\frac{rx}{h}\right)^2\mathrm{d}x = \frac{1}{3}\pi r^2h$
- Sometimes the curve might go below the x axis, but this is fixed by the square
- If the region is bounded by two curves, when rotated about an axis, each slice would be a ring (washer), with area equal to the difference of two circles
- $A_i = \pi\left[(f(x))^2-(g(x))^2\right] \implies V = \int_a^b \pi\left[(f(x))^2-(g(x))^2\right]\mathrm{d}x$, known as the washer method

- If the axis is parallel but not equal to the x axis, then we need to subtract an offset $V = \int_a^b \pi(f(x)-k)^2 \, \mathrm{d}x$ for the disk method, or $V = \int_a^b \pi\left|(f(x)-k)^2 - (g(x)-k)^2\right| \mathrm{d}x$ for the washer method (note the added absolute value)
  - Note the constraint $k > f, g$ or $k < f, g$; the line cannot cross the region itself because then the region would be rotating into itself
- Example: The region between $y = x^2$ and $y = x$ about $y = -2$
  - The two curves intersect at 0 and 1
  - $V = \int_0^1 \pi\left[(x+2)^2 - (x^2+2)^2\right] \mathrm{d}x$

Lecture 23, Nov 3, 2021

Volumes by Cylindrical Shells

- For volumes that may be difficult to integrate with the disk and washer method
- E.g. $y = f(a)$ rotated around the y axis
- Each slice of the area under the curve of f is going to be rotated and forms a hollow cylindrical shell
  - Height: $f(x_i^*)$, radius x_i^*, and thickness Δx; the volume of each shell is the area of the rectangle multiplied by the circumference at that location, $\Delta V = f(x_i^*)\Delta x_i \cdot 2\pi x_i^*$
  - Taking the limit as the number of shells approaches infinity, $V = \int_a^b 2\pi x f(x) \, \mathrm{d}x$
- If the region is defined by the area between two curves then $V = \int_a^b 2\pi x(f(x) - g(x)) \, \mathrm{d}x$
- Similarly we can also rotate around the x axis instead by replacing all the x with y
  - Note that in the washer method, rotation around the x axis leads to integration wrt x, but for the shell method it's the opposite; rotation around the x axis leads to integration wrt y
- Example: Rotate the area between $y^2 - x^2 = 1$ and $y = 2$ about the x axis
  - We can take advantage of symmetry
  - For a given y value, $x = \sqrt{y^2 - 1}$; the minimum y occurs when $x = 0 \implies y = 1$
  - This leads to the integral $V = 2\int_1^2 2\pi y(\sqrt{y^2-1}) \, \mathrm{d}y$
  - To evaluate, substitute $u = y^2 - 1 \implies \mathrm{d}u = 2y \, \mathrm{d}y \implies V = 4\pi \int_0^3 \sqrt{u} \, \mathrm{d}u$

$$= 2\pi \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3$$

$$= 4\sqrt{3}\pi$$

- If rotating about an axis that is parallel to one of the principle axes, subtract an offset $V = \int_a^b 2\pi x(f(x) - k) \, \mathrm{d}x$

Average Value of a Function

- Since the average of a set of discrete numbers is $a_{avg} = \frac{a_1 + a_2 + \cdots + a_n}{n}$, the average of a function taken at discrete points is $f_{avg} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$
- If we consider a uniform partition, then $\Delta x = \frac{b-a}{n} \implies n = \frac{b-a}{\Delta x}$
- If we substitute this into the equation above then $f_{avg} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\Delta x}{b-a}$

- Take the limit $n \to \infty$, the sum turns into an integral $f_{avg} \equiv \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$ (note this is a definition of the average)
- This is defined as the average value of f
- This leads to the *Mean Value Theorem for Integrals*: For f continuous on $[a, b]$, there exists $c \in [a, b]$ such that $f(c) = f_{avg} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$; a continuous function must equal its average value at some point
  - Proof: Let $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$, then by the MVT there exists c such that $F'(c) = \dfrac{F(b) - F(a)}{b-a}$ for $c \in [a, b]$; therefore $F'(c) \implies f(c) = \dfrac{\int_a^b f(t)\,\mathrm{d}t - \int_a^a f(t)\,\mathrm{d}t}{b-a} = \dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,\mathrm{d}t$
  - This is also a special case of the second MVT for $g(x) = 1$
- The *Second Mean Value Theorem* for integrals: For nonnegative g, $\displaystyle\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x$ where $c \in [a, b]$
  - Proof: By the EVT f has a maximum and minimum $m \leq f(x) \leq M$; since g is nonnegative, $mg(x) \leq f(x)g(x) \leq Mg(x) \implies m\displaystyle\int_a^b g(x)\,\mathrm{d}x \leq \int_a^b f(x)g(x)\,\mathrm{d}x \leq M\int_a^b g(x)\,\mathrm{d}x$
    - If g is everywhere zero then the theorem holds true since $0 = 0$
    - Otherwise we can divide by $\displaystyle\int_a^b g(x)\,\mathrm{d}x$ to get $m \leq \dfrac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \leq M$
    - By the IVT, there exists a $c \in [a, b]$ such that $f(c) = \dfrac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}$, since this value is between m and M and f takes on all values in that interval
    - Therefore there exists $c \in [a, b]$ such that $f(c)\displaystyle\int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)g(x)\,\mathrm{d}x$
  - Notes: $f(c)$ is *not* the average of f as from the first MVT; this is a weighted average where $g(x)$ is the weight
  - Think of it like a centre of mass where $g(x)$ describes the density, so $\dfrac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}$ is the weighted average/centre of mass

Lecture 24, Nov 5, 2021

Inverse Functions

- Definition: $f(x)$ is one-to-one if $f(x_1) = f(x_2) \implies x_1 = x_2$; alternatively, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- To check for one-to-one we can use a horizontal line test like the vertical line test for functions
- Definition: Let $f(x)$ be a one-to-one function with domain A and range B, then its *inverse function* $f^{-1}(x)$ has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$, alternatively $f^{-1}(f(x)) = x$
  - Only one-to-one functions possess inverses
- Example: If $f(x) = x^3$, then $y = f^{-1}(x) \implies f(y) = x \implies y^3 = x \implies y = x^{\frac{1}{3}} \implies f^{-1}(x) = \sqrt[3]{x}$
- Functions and their inverses are reflections of each other across $y = x$
- Theorem: If f is either increasing or decreasing then it is one-to-one and hence has an inverse
  - Proof: Suppose $f(x)$ is decreasing, then $x_1 < x_2 \implies f(x_1) > f(x_2)$ and $x_1 > x_2 \implies f(x_1) < f(x_2)$, therefore $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$; same goes for increasing functions
  - Example: $f(x) = 2x - 1$ has $f'(x) = 2 > 0$ therefore $2x - 1$ is one-to-one
  - Note there are functions where the derivative could be equal to zero at *finite* locations but are still increasing or decreasing; e.g. $f(x) = x^3$
- Theorem: If f is continuous, then its inverse is also continuous

- Let $g(x) = f^{-1}(x)$; then $g'(x) = \frac{1}{f'(g(x))}$, or in Leibniz notation, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$
- Example: $f(x) = \frac{1}{x}$ on $(0, \infty)$
  - $f'(x) = -\frac{1}{x^2} < 0$ so the function is decreasing and one-on-one
  - $f^{-1}(x) = \frac{1}{x}$; this function is its own inverse
- The inverse of a composite function $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Natural Logarithms

- Definition: A logarithmic function is a non-constant differentiable function f, defined for $x > 0$, such that for all $a > 0$ and $b > 0$, $f(ab) = f(a) + f(b)$
  - This is all that's required to define logarithms and exponentials!
- We get some properties immediately:
  1. $f(1) = 0$
     - $f(1) = f(1 \cdot 1) = f(1) + f(1) \implies f(1) = 2f(1) \implies f(1) = 0$
  2. $f\left(\frac{1}{x}\right) = -f(x)$
     - $0 = f(1) = f\left(x \cdot \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \implies f\left(\frac{1}{x}\right) = -f(x)$
  3. $f\left(\frac{x}{y}\right) = f(x) - f(y)$
     - $f\left(\frac{x}{y}\right) = f\left(x \cdot \frac{1}{y}\right) = f(x) - f(y)$
  4. $f'(x) = \frac{1}{x} f'(1)$
     - $$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} f\left(\frac{x+h}{x}\right) \\ &= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} \\ &= \frac{1}{x} \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{x \cdot \frac{h}{x}} \\ &= \frac{1}{x} \lim_{k \to 0} \frac{f(1+k) - f(1)}{k} \\ &= \frac{1}{x} f'(1) \end{aligned}$$
- Since the derivative of the logarithm is scaled by $f'(1)$, it's natural to choose $f'(1) = 1$ (note that if we chose 0, then the derivative would be always 0 and thus the function would be constant, violating our constraint), therefore $f'(x) = \frac{1}{x}$; now we can use the FTC to define $f(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t$, starting at 1 because $f(1)$ needs to be zero
- Definition: The natural logarithm, $\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t$ for $x > 0$

Lecture 25, Nov 15, 2021

Properties of $\ln x$

- Further properties of $\ln x$:
  1. Defined on $(0, \infty)$, and $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x}$
  2. $\ln x$ is continuous since it is differentiable
  3. For all $x > 1$, $\ln x > 0$, since the integral area is always positive
  4. For $0 < x < 1$, $\ln x < 0$ as follows from the previous point
  5. $\ln(a + b) = \ln a + \ln b$
     - This can be proven using a different way
     - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x}$, and $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(ax) = \dfrac{1}{ax} \cdot a = \dfrac{1}{x}$
     - Since $\ln x$ and $\ln(ax)$ have same derivative, they differ by a constant, so
     - Therefore $\ln(ax) = \ln(x) + C$, and if we let $x = 1 \implies \ln a = 0 + C = C$, therefore $\ln(ax) = \ln(x) + \ln(a)$
  6. $\ln \left(x^{\frac{p}{q}}\right) = \dfrac{p}{q} \ln x$
     - We can show this in the same way as the one above
     - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln \left(x^{\frac{p}{q}}\right) = \dfrac{1}{x^{\frac{p}{q}}} \cdot \dfrac{p}{q} x^{\frac{p}{q}-1} = \dfrac{p}{q} \dfrac{1}{x} = \dfrac{\mathrm{d}}{\mathrm{d}x} \dfrac{p}{q} \ln x$
     - Therefore $\ln \left(x^{\frac{p}{q}}\right) = \dfrac{p}{q} \ln x + C$, and now if we let $x = 1$, then $0 = C$
  7. The range of $\ln x$ is $(-\infty, \infty)$ (i.e. it is unbounded)
     - Proof that ln is unbounded as $x \to \infty$: Show that for $M > 0$ imposed, there exists a x_0 such that $x > x_0 \implies \ln x > M$
       - Begin with $\ln 2 = \displaystyle\int_1^2 \frac{1}{t}\,\mathrm{d}t$, and since the integrand is always positive, $\ln 2 > 0$
       - Therefore, we can always find some positive n such that $n \ln 2 > M$ no matter how big M is, so choose $x_0 = 2^n$
       - When $x > x_0 = 2^n$, we have $\ln x > n \ln 2 > M$, therefore x is unbounded above
     - A similar argument follows for when $x \to 0$, therefore $x = 0$ is a vertical asymptote
  8. $\ln e = 1$
     - Since the $\ln x$ is unbounded and starts at 0 when $x = 1$, it must take on the value of 1 sometime, so we call this value e
  9. Convention $\ln x = \log_e(x)$
  10. $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x} > 0$ so it is increasing, $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \ln x = -\dfrac{1}{2x} < 0$ so it is concave down

Graphing Logarithms

- Chain rule with logarithms: $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln u = \dfrac{1}{u} \dfrac{\mathrm{d}u}{\mathrm{d}x}$
- When graphing a logarithm note the argument can only be positive, and when it approaches zero, the value of the logarithm approaches negative infinity
- x intercepts when the argument is 1

Using Logarithms to Integrate and Differentiate

- $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$, absolute value because the domain of $\dfrac{1}{x}$ includes the negative numbers
- More generally, $\displaystyle\int \frac{g'(x)}{g(x)}\,\mathrm{d}x = \ln|g(x)| + C$ for $g(x) \neq 0$ (by substitution)

- Example: $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x$
$$\begin{aligned} &= -\int \frac{1}{u} \, \mathrm{d}u \\ &= -\ln|u| \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C \end{aligned}$$
- Example: $\int \cot x \, \mathrm{d}x = \int \frac{\cos x}{\sin x} \, \mathrm{d}x$
$$\begin{aligned} &= \int \frac{1}{u} \, \mathrm{d}u \\ &= \ln|u| + C \\ &= \ln|\sin x| + C \end{aligned}$$
- Example: $\int \sec x \, \mathrm{d}x = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x$
$$\begin{aligned} &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \\ &= \int \frac{1}{u} \, \mathrm{d}u \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C \end{aligned}$$

Lecture 26, Nov 17, 2021

Logarithmic Differentiation

- If we want to differentiate a lengthy product $g(x) = g_1(x)g_2(x) \cdots g_n(x)$, we can take the log of both sides to get $\ln|g(x)| = \ln|g_1(x)| + \ln|g_2(x)| + \cdots + \ln|g_n(x)|$, then differentiate $\frac{g'(x)}{g(x)} = \frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}$, and multiply by $g(x)$ to get $g'(x) = g(x) \sum_{i=1}^{n} \frac{g_i'(x)}{g_i(x)}$ (known as *logarithmic differentiation*)
- Example: $\frac{\mathrm{d}}{\mathrm{d}x} \frac{x^4(x-1)}{(x+2)(x^2+1)} = \frac{x^4(x-1)}{(x+2)(x^2+1)} \left[\frac{4x^3}{x^4} + \frac{1}{x-1} - \frac{1}{x+2} - \frac{2x}{x^2+1} \right]$
  - Notice that if a term is in the denominator it is subtracted instead, since the logarithm of that term is negative

The Natural Exponential

- Using the natural logarithm, we can extend the domain of the exponential to irrational numbers
- From the IVT we know that for some irrational z, $\ln x$ will take on that value at some point
- Definition: Let z be an irrational; then e^z is the unique number such that $\ln e^z = z$
- Definition: The exponential function $\exp x = e^x$, defined as the number such that $\ln e^x = x$
- Properties of the exponential:
  1. ln is the inverse of the exponential: $\ln e^x = x$ for $x \in \mathbb{R}$ as per our definition of the exponential and irrational powers
  2. $e^x > 0$, which comes from the fact that $\ln e^x$ is only defined for positive e^x
  3. $e^0 = 1$ from $\ln 1 = 0$
  4. $\lim_{x \to -\infty} e^x = 0$
  5. $e^{\ln x} = x$
  6. $e^{a+b} = e^a \cdot e^b$
     - From $\ln(e^a \cdot e^b) = \ln e^a + \ln e^b = a + b = \ln e^{a+b}$

7. In a similar manner $e^{-b} = \frac{1}{e^b}$ and $e^{a-b} = \frac{e^a}{e^b}$, both of which come from the logarithm
8. $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$
   - $\ln e^x = x \implies \frac{\mathrm{d}}{\mathrm{d}x} \ln e^x = \frac{1}{e^x} \frac{\mathrm{d}}{\mathrm{d}x} e^x = \frac{\mathrm{d}}{\mathrm{d}x} x = 1 \implies \frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$
9. $\frac{\mathrm{d}}{\mathrm{d}x} e^{kx} = ke^{kx}$ from the chain rule
10. $\int e^x \, \mathrm{d}x = e^x + C$
11. $\int e^{g(x)} g'(x) \, \mathrm{d}x = e^{g(x)} + C$

Lecture 27, Nov 29, 2021

Approximation of e^x

- Example: Show $e^x > 1 + x$ for $x > 0$
  - First show that $e^x > 1$ using integrals, and then take this back into the integral and show that $e^x > 1 + x$, then repeat this more to get $e^x > 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$
  - Key identity: $e^x = 1 + \int_0^x e^t \, \mathrm{d}t$
  - Note that $e^0 = 1$ and $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x > 0$ so it is always positive and increasing, and $e^x > 1$ for $x > 0$
  - $\int_0^x e^t \, \mathrm{d}t > \int_0^x \, \mathrm{d}t \implies 1 + \int_0^x e^t \, \mathrm{d}t > \int_0^x \, \mathrm{d}t + 1 \implies e^x > 1 + x$
  - We can continue this process making use of $e^x > 1 + x$, so $e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x (1 + t) \, \mathrm{d}t = 1 + x + \frac{x^2}{2}$
  - We can do this again one more time: $e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x (1 + t + \frac{t^2}{2}) \, \mathrm{d}t = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
  - Using induction we can show $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
    * Note we don't yet know that this infinite series converges to e

General Exponential Function

- Definition: An irrational power $x^z = e^{z \ln x}$
  - Thus with this extended definition we have $x^{r+x} = x^r x^s$, $x^{r-s} = \frac{x^r}{x^s}$, $(x^r)^s = x^{rs}$ for $r, s \in \mathbb{R}$ provided $x > 0$
- We can extend the power rule to irrational powers
- Proof: $\frac{\mathrm{d}}{\mathrm{d}x} x^r = \frac{\mathrm{d}}{\mathrm{d}x} e^{r \ln x} = e^{r \ln x} \cdot \frac{r}{x} = \frac{x^r \cdot r}{x} = rx^{r-1}$, and with this we can also extend the reverse power rule for integrals
- Since x^z is defined using the natural exponential, it has the same properties
- Example: $\frac{\mathrm{d}}{\mathrm{d}x} x^x$
  - $x^x = e^{x \ln x} \implies \frac{\mathrm{d}}{\mathrm{d}x} x^x = e^{x \ln x} \left(x \cdot \frac{1}{x} + \ln x \right) = x^x (1 + \ln x)$
- Exponentials with bases other than p: $\frac{\mathrm{d}}{\mathrm{d}x} p^x$
  - $\frac{\mathrm{d}}{\mathrm{d}x} p^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x \ln p} = \ln(p) e^{x \ln p} = \ln(p) p^x$
- In general $\frac{\mathrm{d}}{\mathrm{d}x} p^u = p^u \ln(p) \frac{\mathrm{d}u}{\mathrm{d}x}$

- The integral form is $\int p^x \, \mathrm{d}x = \frac{1}{\ln p} p^x + C$ where $p > 0, p \neq 1$

Logarithm With Other Bases

- Define $f(x) = \frac{\ln x}{\ln p}$ and $g(x) = p^x$, then $f(g(x)) = \frac{\ln p^x}{\ln p} = x\frac{\ln p}{\ln p} = x$ and $g(f(x)) = p^{\left(\frac{\ln x}{\ln p}\right)} = e^{\frac{\ln x}{\ln p} \cdot \ln p} = e^{\ln x} = x$, so they are inverses
- Define $\log_p(x) = \frac{\ln x}{\ln p}$ for $p > 0, p \neq 1$, note $\log_p(p^x) = x$ as they are inverses of each other
- $\frac{\mathrm{d}}{\mathrm{d}x} \log_p(u) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln u}{\ln p} = \frac{1}{\ln p} \cdot \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$

Estimating e

- $\ln x = \int_1^x \frac{\mathrm{d}t}{t} \implies \ln\left(1 + \frac{1}{n}\right) = \int_1^{1+\frac{1}{n}} \frac{\mathrm{d}t}{t} < \int_1^{1+\frac{1}{n}} = 1 + \frac{1}{n} - 1 = \frac{1}{n}$
- $\ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \implies 1 + \frac{1}{n} < e^{\frac{1}{n}} \implies \left(1 + \frac{1}{n}\right)^n < e$
- To find an upper bound, $\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+\frac{1}{n}} \frac{1}{t} \, \mathrm{d}t > \int_1^{1+\frac{1}{n}} \frac{1}{1 + \frac{1}{n}} \, \mathrm{d}t$, since $1 < t < \frac{1}{1 + \frac{1}{n}}$
- Therefore $\ln\left(1 + \frac{1}{n}\right) > \left(\frac{1}{1 + \frac{1}{n}}\right)\left(1 + \frac{1}{n} - 1\right) = \frac{1}{n+1} \implies \left(1 + \frac{1}{n}\right)^{n+1} > e$
- Combining these two we get $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$
- As n tends to infinity, the difference between n and $n + 1$ becomes small and the two bounds close in on each other to converge to the true value of e
- $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ by a squeeze-theorem like argument

Lecture 28, Nov 22, 2021

Inverse Trigonometric Functions

- Since trig functions are not one-to-one, we cannot find general inverses, but they are invertible if we restrict their domains
- $\sin^{-1} x$ or $\arcsin x$ has domain $x \in [-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\sin^{-1}(\sin x) = x$ for $x \in [-1, 1]$
  - $\sin(\sin^{-1} x) = x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - Note the inverse sine is an odd function
- To find the derivative: $\frac{\mathrm{d}}{\mathrm{d}x} \sin(\sin^{-1} x) = \frac{\mathrm{d}}{\mathrm{d}x} x \implies \frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - x^2}}$
  - Note $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ is a relationship that can be demonstrated geometrically
  - Consider a triangle with hypotenuse 1 and angle $\theta = \sin^{-1} x$, then $\sin\theta = x$ which is the opposite side, and $\cos\theta$ is the adjacent side; so by the Pythagorean theorem, $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1 - x^2}$
- Reversing this we get $\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + C$ (using u substitution with $au = x$
- Inverse tangent $\tan^{-1} x = \arctan x$ has domain $x \in (-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - $\tan(\tan^{-1} x) = x$ for all x but $\tan^{-1}(\tan x) = x$ only for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - Note the period of tan is π, not 2π
- The derivative $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + x^2}$

- $\sec^2(\tan^{-1} x) = 1 + x^2$ can be derived from a similar geometric process for sin, with the adjacent side 1 and opposite side x
- $\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- The inverse secant $\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}$, so $\int \frac{1}{x\sqrt{x^2 - a^2}}\,\mathrm{d}x = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Modelling with Differential Equations

- Often it's much easier to model something against its rate of change; e.g. the rate of change of heat (heat loss) is proportional to the difference in temperature
- Define a *differential equation* with an equation which contains an unknown function with one or more of its derivatives
  - Example: $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$ or $F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ or $\nabla \times \vec{E} = -\dot{\vec{B}}$
- Differential equations can be classified as ordinary (1 independent variable) or partial (multiple independent variables), linear (only scalar multiples of derivatives) or nonlinear, and its order is the order of the highest derivative

Lecture 29, Nov 24, 2021

Solutions to Differential Equations

- Definition: A function is a solution of a DE if the substitution of the function and its derivatives lead to an identity.
  - Example: $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = 2x + \frac{4}{x}$ has solution $y(x) = 2x^2 + 7x - 4$
- The basic integration process is like a very simple differential equation; $\frac{\mathrm{d}y}{\mathrm{d}x}f(x) \implies y(x) = F(x) + C$
  - There is not one unique solution to a differential equation; generally the number of free parameters in the solution to a DE is equal to its order (e.g. first order DEs have 1 constant, second order DEs have 2)
  - The values of these constants can be determined by substituting in known values from initial conditions
- The *general solution* to an n-th order DE is an n-parameter family of solutions that include all the solutions to the DE.
- The *particular solution* is a member of the general solutions family for the DE and have specific values assigned to the constants from initial values
- An *initial value problem* consists of a DE and a number of initial values for the function; a *boundary value problem* is a DE and some known values not at 0

Separable Differential Equations

- Not all differential equations can be solved analytically, but some are easier to solve than others
- A general first-order DE can be expressed as $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y)$; if $F(x, y)$ can be separated into $F(x, y) = g(x)f(y)$ or $\frac{g(x)}{h(y)}$, then it is *separable* and can be solved easily
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)} \implies \int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$
    - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)} \implies h(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \implies \int h(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x \implies \int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$

* Note: Assume $\frac{\mathrm{d}}{\mathrm{d}y}H(y) = h(y)$ then $\frac{\mathrm{d}}{\mathrm{d}x}H(y) = h(y)\frac{\mathrm{d}y}{\mathrm{d}x}$ by the chain rule, which means that $\int h(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}H(y)\,\mathrm{d}y = H(y) = \int h(y)\,\mathrm{d}y$

- Example: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}e^x y^2$ with $y(0) = -1$
  - $\frac{1}{2}e^x y^2 = \frac{e^x}{\frac{2}{y^2}} \implies \int \frac{2}{y^2}\,\mathrm{d}y = \int e^x\,\mathrm{d}x \implies -2y^{-1} = e^x + C \implies y = -\frac{2}{e^x + C}$ is the general solution
  - Use the initial condition gets us $y = -\frac{2}{1+C} = -1 \implies C = 1$ so the particular solution is $y = -\frac{2}{e^x + 1}$
  - Note this equation is nonlinear
- Example: Resistor-Inductor (RL) circuits
  - Suppose there is a resistor I and inductor L connected in series to a supply voltage of $E(t)$, then by Ohm's law the voltage drop across the resistor is $V = RI$ and the voltage drop across the inductor is $V = L\frac{\mathrm{d}I}{\mathrm{d}t}$, so $E(t) = L\frac{\mathrm{d}I}{\mathrm{d}t} + RI$ models the current I
  - Rewrite as $\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{V - RI}{L} \implies g(x) = 1, h(y) = \frac{L}{V - RI} \implies \int \frac{L}{V - RI}\,\mathrm{d}I = \int \mathrm{d}t$
  - After substituting in the numbers we can easily integrate and find a solution

Orthogonal Trajectories

- Given a family of curves, an *orthogonal trajectory* is a trajectory that passes through every one of these curves at 90 degrees
- The tangent of the trajectory is everywhere orthogonal to the tangent of the curves, so $f' = -\frac{1}{g'}$
- Example: $y^2 = kx^3$
  - Using implicit differentiation, $2yy' = 3kx^2 \implies y' = \frac{3kx^2}{2y}$
  - To get the general y', substitute back $y^2 = kx^3 \implies k = \frac{y^2}{x^3} \implies y' = \frac{3\left(\frac{y^2}{x^3}\right)x^2}{2y} = \frac{3}{2}\frac{y}{x}$
  - Then the tangent to the curve we want is $y' = \frac{-2x}{3y}$ and this is a separable differential equation so $\int 3y\,\mathrm{d}y = -\frac{2x}{}\mathrm{d}x \implies \frac{3y^2}{2} = -x^2 + C \implies 3y^2 + 2x^2 = 2C$
  - This is a family of ellipses

Lecture 30, Nov 26, 2021

Exponential Growth and Decay

- When a quantity changes at a rate proportional to the quantity itself, $\frac{\mathrm{d}f}{\mathrm{d}t} = kf(t)$, and this leads to exponential growth or decay of $f(t) = Ce^{kt}$ where C is the initial condition
  - $k = \frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}t}$, using the chain rule backwards this is equal to $\frac{\mathrm{d}}{\mathrm{d}t}\ln f$
  - Integrating both sides, $\ln f = kt + C \implies f = e^{kt+c} = Ce^{kt}$
  - C is the initial value since $f(0) = Ce^0 = C$
  - k is the growth or decay constant
- We can also characterize exponential growth by the doubling time: $2P_0 = P_0e^{kt_2} \implies t_2 = \frac{\ln 2}{k}$

Radioactive Decay

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -kN$ where k is always a positive
- $N(t) = N(0)e^{-kt}$
- Here we use the half-life and it basically works the exact same way with $t_{\frac{1}{2}} = \frac{\ln 2}{k}$
- Example: A year ago we had 4kg of a radioactive material; now we have 3kg. How much did we have 10 years ago?
  - $t = 0$ 10 years ago, therefore $4 = N_0 e^{-9k}$ and $3 = N_0 e^{-10k}$
  - Dividing the equations we get $\frac{4}{3} = e^{k(-9+10)} = e^k \implies k = \ln\frac{4}{3} \approx 0.288$
  - $N_0 = 4e^{9k} = 53.3\text{kg}$
  - The half life is $\frac{\ln 2}{k} = 2.4$ years

Compound Interest

- If interest is compounded at fixed intervals, then $V(t) = V(0)(1+i)^t$ where i is the interest rate
  - If we shorten this interval by n times then $V(t) = V(0)\left(1 + \frac{i}{n}\right)^{tn}$
  - $\lim_{n\to\infty}\left(1+\frac{i}{n}\right)^{nt} = \lim_{n\to\infty}\left(1 + \frac{1}{\frac{n}{i}}\right)^{\frac{n}{i}it}$, with the substitution $m = \frac{n}{i}$ it becomes $\lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^{it} = e^{it}$
  - Therefore at the maximum rate of compounding, $V(t) = V(0)e^{it}$

Drug Metabolism

- Drug metabolism can also be modelled as the rate of elimination being proportional to the current concentration, which leads to exponential decay $C_0 e^{-kt}$
- Typically we want to maintain the concentration of the drug in the blood between some therapeutic level and some other toxic level
- Using this model we can time the injection of the drugs so it stays between the two levels

Population Growth: The Logistic Model

- $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is not a very accurate model of the population growth because it implies that the population grows exponentially without bound
- Usually population growth tends to approach 0 as the population reaches some carrying capacity due to various factors
- The *logistic model* models population as $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$ with M as the carrying capacity or max population
  - As P approaches M the growth slows down, and when $P = M$, $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ and the population stops growing
- Integrating both sides: $\int \frac{1}{P\left(1 - \frac{P}{M}\right)}\,\mathrm{d}t = k\int \mathrm{d}t$
  - Note that $\frac{1}{P\left(1-\frac{P}{M}\right)} = \frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$

- $$\int \frac{1}{P\left(1-\frac{P}{M}\right)}\,\mathrm{d}t = k\int \mathrm{d}t$$
  $$\implies \int \frac{1}{P} + \frac{1}{M-P}\,\mathrm{d}P = \ln|P| - \ln|M-P| = kt + C$$
  $$\implies \ln\left|\frac{P}{M-P}\right| = kt + C$$
  $$\implies \frac{P}{M-P} = \pm e^{kt+c}$$
  $$\implies \frac{M-P}{P} = Ae^{-kt}$$
  $$\implies P(t) = \frac{M}{1+Ae^{-kt}}$$
  - Where $A = \dfrac{M-P_0}{P_0}$ and $P_0 = P(0)$
- When t is small, the growth is exponential, and then the growth slows down and approaches M exponentially

Linear Equations

- A first-order linear ODE can be written in the form of $y' + p(x)y = q(x)$
- All first order linear ODEs have a general solution
- Example: $xy' + y = x^2$
  - Writing this in the standard form: $y' + \dfrac{1}{x}y = x$ for $x \neq 0$
  - The left hand side is the product rule applied to xy: $(xy)' = xy' + y$
  - So the equation becomes $(xy)' = x^2 \implies \displaystyle\int (xy)'\,\mathrm{d}x = \int x^2\,\mathrm{d}x \implies xy = \frac{x^3}{3} + C \implies y = \frac{x^2}{3} + \frac{C}{x}$
  - In general, we want to turn the left hand side into a product rule expression
- To set up the general case, set up $H(x) = \displaystyle\int p(x)\,\mathrm{d}x$ (don't worry about constants for this)
- Therefore $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{H(x)} = p(x)e^{H(x)}$
- Putting this back into the equation: $\dfrac{\mathrm{d}}{\mathrm{d}x}ye^{H(x)} = y'e^{H(x)} + ye^{H(x)}p(x) = e^{H(x)}(y' + p(x)y)$, and $y' + p(x)y$ is just the left hand side of our equation, so it equals $q(x)$
  - $e^{H(x)}$ is known as the *integrating factor*, and by multiplying the equation through by this factor, we end up with $\dfrac{\mathrm{d}}{\mathrm{d}x}ye^{H(x)} = e^{H(x)}q(x)$
- $ye^{H(x)} = \displaystyle\int e^{H(x)}q(x)\,\mathrm{d}x + C$, so our final answer is $y = e^{-H(x)}\left(\displaystyle\int e^{H(x)}q(x)\,\mathrm{d}x + C\right)$
  - Usually the constant of integration is put in at the first stage there so that we don't forget about it
- To solve these equations:
  1. Write the equation explicitly in the form of $y' + p(x)y = q(x)$ and determine $p(x)$ and $q(x)$
  2. Find the integrating factor $e^{H(x)} = e^{\int p(x)\,\mathrm{d}x}$
  3. Multiply the equation by the integrating factor
  4. Integrate both sides
  5. Isolate for y
- Example: $y' + 2y = 4 \implies p(x) = 2, q(x) = 4$, so the integrating factor is e^{2x}, and $e^{2x}y' + 2e^{2x}y = 4e^{2x} \implies \dfrac{\mathrm{d}}{\mathrm{d}x}(e^{2x}y) = 4e^{2x} \implies e^{2x}y = 4\displaystyle\int e^{2x}\,\mathrm{d}x + C = 2e^{2x} + C$ so the final answer is $y = 2 + Ce^{-2x}$
  - We can see that the solution can be separated into 2 parts, one part as a particular solution ($y = 2$), and the other for solving $y' + 2y = 0$ – this will come back in second order linear ODEs
- Example: Newton's law of cooling

- $\dfrac{\mathrm{d}T}{\mathrm{d}t} = -k(T-\tau)$, the change in temperature is proportional to the difference in temperature between the object and its surroundings
  - Note the negative sign indicates that if the object is hotter than its surroundings then its temperature will decrease
- $T' + kT = k\tau \implies p(t) = k, q(t) = k\tau$
- Integrating factor $e^{H(t)} = e^{kt}$
- $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(e^{kt} + T\right) = e^{kT}k\tau$
- $T = e^{-kt}\left(\displaystyle\int e^{kt}k\tau\,\mathrm{d}t + C\right) = \tau + Ce^{-kt}$

- To summarize, $y' + p(x)y' = q(x)$ has solution $y = e^{-\int p(x)\,\mathrm{d}x}\left[\displaystyle\int e^{\int p(x)\,\mathrm{d}x}q(x)\,\mathrm{d}x + C\right]$

Lecture 31, Nov 29, 2021

Linear Equations Continued

- Example: Logistic model $\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \dfrac{P}{M}\right)$ with $M(t)$ and $k(t)$ as a function of time
  - First we have to make the equation linear since there is a P^2 term; substitute $z = \dfrac{1}{p} \implies P = \dfrac{1}{z} \implies \dfrac{\mathrm{d}P}{\mathrm{d}t} = -\dfrac{1}{z^2}\dfrac{\mathrm{d}z}{\mathrm{d}t}$
  - $-\dfrac{1}{z^2}\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{k}{z}\left(1 - \dfrac{1}{ZM}\right) \implies \dfrac{\mathrm{d}z}{\mathrm{d}t} = -kz\left(1 - \dfrac{1}{ZM}\right) \implies \dfrac{\mathrm{d}z}{\mathrm{d}t} + kz = \dfrac{k}{M}$ and it is now linear
  - If k and M are constant, then integrating factor: $H(t) = kt \implies$ integrating factor is e^{kt}
  - $z = e^{-kt}\left[\displaystyle\int e^{kt}\frac{k}{M}\,\mathrm{d}t + C\right] = e^{-kt}\left[\dfrac{e^{kt}}{M} + C\right] = \dfrac{1}{M} + Ce^{-kt}$
  - Therefore $P = \dfrac{1}{z} = \dfrac{M}{1 + MCe^{-kt}}$
- Example: Solving an RC (resistor-capacitor) circuit
  - Voltage drop across a capacitor is $\dfrac{Q}{C}$ where Q is the charge in coulombs and C is capacitance in Farads
  - $E(t) = \dfrac{Q}{C} = IR$, but since $I = \dfrac{\mathrm{d}Q}{\mathrm{d}t}$, $E(t) = \dfrac{Q}{C} + R\dfrac{\mathrm{d}Q}{\mathrm{d}t}$
  - In standard form, $Q' + \dfrac{1}{RC}Q = \dfrac{E(t)}{R}$
  - $H(t) = \displaystyle\int \frac{1}{RC}\,\mathrm{d}t = \frac{t}{RC}$ so the integrating factor is $e^{\frac{t}{RC}}$
  - $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{t}{RC}}Q\right) = e^{\frac{t}{RC}}\dfrac{E(t)}{R}$
  - $Q = e^{-\frac{t}{RC}}\left(\displaystyle\int e^{\frac{t}{RC}}\frac{E(t)}{R}\,\mathrm{d}t + A\right)$
- A nonlinear equation in the form $y' + p(x)y = q(x)y^r$ can be made linear by substituting $u = y^{1-r}$
  - $u' = (1-r)y^{-r}$
  - $y' + p(x)y = q(x)y^r \implies (1-r)y^{-r}y' + (1-r)y^{-r}y = (1-r)y^{-r}q(x)y^r \implies u' + (1-r)p(x)u = (1-r)q(x)$
  - Now we can apply the integrating factor
  - Equations in this form are called *Bernoulli Equations*

Complex Numbers

- Our number system is missing solutions to equations such as $x^2 = -1$, and like we extended the reals with CORA, we need more axioms to extend our number system

- Calculus of complex number systems is called complex analysis
- Define the *imaginary unit* $i^2 = -1$ or $i = \sqrt{-1}$, and a *complex number* $z = a + ib$ where $a, b \in \mathbb{R}$ and $\text{Re}(z) = a, \text{Im}(z) = b$
- We can represent the complex number $a + ib$ as the point (a, b) on the complex plane (instead of a number line), where the horizontal axis is the real axis and the vertical axis is the imaginary axis
  - This is known as an *Argand Diagram*, and the plane is known as the complex plane, $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$
- Complex numbers can be represented in a polar form (like polar vectors)
  - The distance from the origin of a complex number is known as the *modulus*: $|z| = |a+ib| = \sqrt{a^2 + b^2}$
    * The modulus of a complex number is like the absolute value of a real number, and it is always real and nonnegative
  - The angle that the complex number makes with the real axis is the known as the *argument*: $\arg(z) = \theta$
    * Arguments are not unique; $\arg(z) = \theta \implies \arg(z) = \theta + 2k\pi$ where $k \in \mathbb{Z}$
  - $|z|\cos(\arg(z)) = a$, $|z|\sin(\arg(z)) = b$, and $\tan(\arg(z)) = \dfrac{b}{a}$ for $a \neq 0$
  - Let $r = |z|$ and $\theta = \arg(z)$, then the polar representation of z is $z = r\cos\theta + ir\sin\theta$
- Definition: The complex conjugate of $z = a + ib$ is $\bar{z} = a - ib$
  - In an Argand diagram the conjugate is a reflection across the real axis

Lecture 32, Dec 1, 2021

Complex Arithmetic

- Complex addition: $z = a + ib, w = c + id \implies z + w = (a + c) + i(b + d), z - w = (a - c) + i(b - d)$
  - Properties are the same as regular addition:
    1. Commutative $z_1 + z_2 = z_2 + z_1$
    2. Associative $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
    3. Triangle inequality $|z_1 \pm z_2| \leq |z_1| \pm |z_2|$
  - The conjugate of the sum is the sum of the conjugate: $\overline{z + w} = \bar{z} + \bar{w}$
- Complex multiplication: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$
  - Product of a complex number and its conjugate is the square of the modulus: $z\bar{z} = |z|^2$
  - Properties are like regular multiplication:
    1. Commutative
    2. Associative
  - $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
  - In polar form: Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$
    * $$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= r_1 r_2((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)) \\ &= r_1 r_2(\cos(\theta + \phi) + i\sin(\theta + \phi)) \end{aligned}$$
    * Therefore $|z_1 z_2| = |z_1||z_2|$, and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
    * Graphically, a product multiplies the lengths and adds the angles
    * Generally, if we have multiple products, the modulus of the product is the product of the moduli and the argument of the product is the sum of the arguments
      - Example: Multiplication by $z = i$ is a counterclockwise rotation by 90° but no change in the modulus
- Division/reciprocals: $z^{-1} = \dfrac{1}{z} = \dfrac{1}{a + ib} = \dfrac{a - ib}{(a + ib)(a - ib)} = \dfrac{a - ib}{a^2 + b^2} = \dfrac{\bar{z}}{|z|^2}$
  - Since $|\bar{z}| = |z|$ and $\arg(\bar{z}) = -\arg(z)$, we have $\left|\dfrac{1}{z}\right| = \dfrac{|\bar{z}|}{|z|^2} = \dfrac{1}{|z|}$ and $\arg\left(\dfrac{1}{z}\right) = \arg\left(\bar{z}\dfrac{1}{|z|^2}\right) = -\arg(z)$
  - Define the complex quotient $\dfrac{z}{w} = zw^{-1} = \dfrac{z\bar{w}}{|w|^2} = \dfrac{(a + ib)(c - id)}{c^2 + d^2} = \dfrac{(ac + bd) + i(ad - cb)}{c^2 + d^2}$

- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
- $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$

De Moivre's Theorem

- Let $z = \cos\theta + i\sin\theta$, then $|z| = 1$ and $\arg z = \theta$
- Then $|z^n| = |z|^n = 1$ and $\arg(z^n) = n\arg(z) = n\theta$
- Therefore $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Complex Exponentials

- To define e^{ix} we can no longer take the log approach because $ix \notin$ the range of ln
- Let $f(x) = e^{ix}$ and $g(x) = \cos x + i\sin x$, then $f'(x) = ie^{ix} = if(x)$ and $g'(x) = -\sin x + i\cos x = ig(x)$, and also $f(0) = e^0 = i$ and $g(0) = \cos 0 + i\sin 0 = 1$
  - Euler therefore concluded that $f(x) = g(x)$ so $e^{ix} = \cos x + i\sin x$
  - Note this is not a sufficient proof since we haven't proven the complex power rule, so this is more of a definition than a proof
- Therefore $e^z = e^{a+ib} = e^a(\cos b + i\sin b)$

Second Order Linear Differential Equations

- A general second order linear ODE has the form $p(x)y'' + q(x)y' + r(x)y = g(x)$
- To begin we look at second linear DEs with constant coefficients: $y'' + ay' + by = g(x)$, and start where $g(x) = 0$, known as a *homogeneous* second order linear DE with constant coefficients
- Theorem: If $y_1(x)$ and $y_2(x)$ are both solutions of a homogeneous second order linear differential equation, then any linear combination $c_1y_1 + c_2y_2$ is also a solution
  - Proof: $(c_1y_1 + c_2y_2)'' + a(c_1y_1 + c_2y_2)' + b(c_1y_1 + c_2y_2) = c_1(y_1'' + ay_1' + by_1) + c_2(y_2'' + ay_2' + by_2) = 0 + 0 = 0$
- Theorem: If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a homogeneous second order linear differential equation, then $c_1y_1 + c_2y_2$ is the general solution
  - Linearly independent means that $y_2 \neq cy_1$
  - Proof is more involved and is covered in a later course
- We can try the solution $y = e^{rx} \implies (e^{rx})'' + a(e^{rx})' + be^{rx} = 0 \implies r^2e^{rx} + are^{rx} + be^{rx} = 0 \implies (r^2 + ar + b)e^{rx} = 0$
  - Thus, $y = e^{rx}$ is a solution if r is a root of $r^2 + ar + b$
  - $r^2 + ar + b = 0$ is known as the *characteristic* or *auxiliary equation* of this differential equation

Lecture 33, Dec 3, 2021

Second Order Linear Homogeneous ODEs With Constant Coefficients: Cases

- Depending on the discriminant of the auxiliary equation, 3 cases can occur:
  1. $a^2 - 4b > 0$ – Two real and distinct roots, so the solutions are e^{r_1x} and e^{r_2x} and we have a general solution $y = c_1e^{r_1x} + c_2e^{r_2x}$
  2. $a^2 - 4b = 0$ – Two equal roots, so one solution is $y_1 = e^{rx}$
     - The other solution turns out to be $y_2 = xe^{rx}$
     - Plugging in y_2 gets us $(xr^2e^{rx} + re^{rx} + re^{rx}) + a(xre^{rx} + e^{rx}) + bxe^{rx} = (r^2 + ar + b)xe^{rx} + (2r + a)e^{rx} = 0$
  3. $a^2 - 4b < 0$ – Two complex roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

- $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$
  $= c_1 e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + c_2 e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$
  $= e^{\alpha x}((c_1+c_2)\cos(\beta x) + i(c_1-c_2)\sin(\beta x))$
  $= e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$

- To evaluate the constants we need known values
  - For an IVP this is the values of $y(0)$ and $y'(0)$, and for a boundary value problem this may be y at two values, y' at two values, or anything else
    - "You can't always fit a square peg into a round hole"
  - IVPs in this case always have unique solutions, while boundary value problems don't
- Example IVP: $y'' + y' - 2y = 0$, $y(0) = 2, y'(0) = 5$
  - Characteristic equation: $r^2 + r - 2 = 0 \implies r_1 = -2, r_2 = 1 \implies y = C_1 e^x + C_2 e^{-2x}$
    - $y' = C_1 e^x - 2C_2 e^{-2x}$
  - Plugging in initial values: $\begin{cases} 2 = C_1 + C_2 \\ 5 = C_1 - 2C_2 \end{cases} \implies \begin{cases} C_1 = 3 \\ C_2 = -1 \end{cases}$
  - $y = 3e^x - e^{-2x}$
- Example BVP: $y'' + 4y' + 5y = 0$, $y(0) = 1, y\left(\frac{\pi}{2}\right) = 0$
  - Characteristic equation: $r^2 + 4r + 5 = 0 \implies r = -2 \pm i \implies y = e^{-2x}(A\cos x + B\sin x)$
  - Apply boundary conditions: $\begin{cases} 1 = 1(A \cdot 1 + B \cdot 0) = A \implies A = 1 \\ 0 = e^{-\pi}(A \cdot 0 + B \cdot 1) = Be^{-\pi} \implies B = 0 \end{cases}$
  - $y = e^{-2x}\cos x$
  - If we change the boundary conditions to $y(0) = 1, y(\pi) = 0$, then the second equation becomes $0 = e^{-2\pi}(-A + B \cdot 0) = -Ae^{-2\pi} \implies A = 0$, but from the first equation $A = 1$, therefore these boundary conditions make the equation unsolvable

Nonhomogeneous Linear ODEs (Constant Coefficients)

- $y'' + ay' + by = \phi(x)$ is a nonhomogeneous linear ODE with constant coefficients
  - Its associated homogeneous linear ODE, the *complementary equation*, is $y'' + ay' + by = 0$
- Theorem: The general solution to a nonhomogeneous second order linear ODE with constant coefficients is $y(x) = y_p(x) + y_c(x)$ where $y_p(x)$ is a particular solution and $y_c(x)$ is the solution of the complementary equation
  - Proof: Given $y_{p_1}(x)$ and $y_{p_2}(x)$, we can show that the latter is a linear combination of the first and $y_c(x)$
    - Let $z = y_{p_1} - y_{p_2}$, if we can show that z is a solution to the complementary solution then we can prove this
    - Since $y_{p_2} = y_{p_1} - z \implies y'_{p_2} = y'_{p_1} - z'$ and $y''_{p_2} = y''_{p_1} - z''$
    - $$\begin{aligned} & y''_{p_2} + ay'_{p_2} + by_{p_2} = \phi(x) \\ \implies & (y''_{p_1} - z'') + a(y'_{p_1} - z') + b(y_{p_1} - z) = \phi(x) \\ \implies & (y''_{p_1} + ay'_{p_1} + by_{p_1}) - (z'' + az' + bz) = \phi(x) \\ \implies & z'' + az' + bz = (y''_{p_1} + ay'_{p_1} + by_{p_1}) - \phi(x) \\ \implies & z'' + az' + bz = 0 \end{aligned}$$
    - Therefore z is a solution to the complementary equation; thus any solution is a linear combination of one particular solution and the complementary solution
  - Thus we only need to find one particular solution to the nonhomogeneous equation, and then add on the solution to the complementary equation to obtain all solutions
- To find y_p, we can use one of the methods below:

Method of Undetermined Coefficients

- This method is easier to do but more limited

- Assume that y_p has the same form as $\phi(x)$, take derivatives and substitute, and then solve for the undetermined coefficients
- Example: $y'' - 6y' + 8y = x^2 + 2x$
  - Complementary equation is $y'' - 6y' + 8y = 0$, solving the auxiliary equation gets us $r_1 = 2, r_2 = 4$ so $y_c = C_1e^{2x} + C_2e^{4x}$
  - Here $\phi(x)$ is a second order polynomial so try y_p in the same form, so $y_p = Ax^2 + Bx + C$
    - $y_p' = 2Ax + B$, $y_p'' = 2A$
  - Substitute the trial solution: $2A - 6(2Ax + B) + 8(Ax^2 + Bx + C) = x^2 + 2x \implies 8Ax^2 + (-12A + 8B)x + (2A - 6B + 8C) = x^2 + 2x$
    - From this we know $\begin{cases} 8A = 1 \\ -12A + 8B = 2 \\ 2A - 6B + 8C = 0 \end{cases}$
    - We can solve for $A = \frac{1}{8}, B = \frac{7}{16}, C = \frac{19}{64}$
  - $y_p = \frac{1}{8}x^2 + \frac{7}{16}x + \frac{19}{64}$
  - $y = \frac{1}{8}x^2 + \frac{7}{16}x + \frac{19}{64} + C_1e^{2x} + C_2e^{4x}$
- Examples of trial solutions:
  - $\phi(x) = e^{3x} \implies y_p = Ae^{3x}$
  - $\phi(x) = C\cos(kx)$ *or* $C\sin(kx)$ then $y_p = A\cos(kx) + B\sin(kx)$
  - $\phi(x) = x^2\sin(kx) \implies y_p = (Ax^2 + Bx + C)\cos(kx) + (Dx^2 + Ex + F)\sin(kx)$
- If $\phi(x)$ is a sum, then we can use the superposition principle: If $y'' + ay' + by = \phi_1(x) + \phi_2(x)$, then $\begin{cases} y_{p_1}'' + ay_{p_1}' + by_{p_1} = \phi_1(x) \\ y_{p_2}'' + ay_{p_2}' + by_{p_2} = \phi_2(x) \end{cases} \implies y_p = y_{p_1} + y_{p_2}$
- If the obvious trial solution is a multiple of y_c, then multiply y_p by x
  - Example: $y'' + y = \sin x$ has $y_c = C_1\cos x + C_2\sin x$, so try $y_p = Ax\cos x + Bx\sin x$
- Example: $y'' - y' - 6y = e^{-2x}$
  - $r^2 - r - 6 = 0 \implies r_1 = -2, r_2 = 3 \implies y_c = C_1e^{-2x} + C_2e^{3x}$
  - Try trial solution $y_p = Axe^{-2x} \implies y_p' = Ae^{-2x} - 2Axe^{-2x} \implies y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x} = (4Ax - 4A)e^{-2x}$
  - $(4Ax - 4A)e^{-2x} - (A - 2Ax) - 6Axe^{-2x} = e^{-2x} \implies \begin{cases} 4A + 2A - 6A = 0 \\ -4A - A = 1 \end{cases} \implies A = -\frac{1}{5}$
  - Therefore the general solution is $y = C_1e^{-2x} + C_2e^{3x} - \frac{1}{5}xe^{-2x}$
- This method guesses for the answer so it's not always possible to solve for the coefficients

Lecture 34, Dec 6, 2021

Variation of Parameters

- A more robust method of finding the particular solution
- The complementary solution is $y_c = C_1y_1(x) + C_2y_2(x)$; to begin this method, we try allowing these constants C_1 and C_2 to vary
- Let the particular solution $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$, now we solve for u_1 and u_2
  - To fully specify these functions we need 2 conditions; the first one will be the differential equation, and the second one will be chosen to make things convenient
- $y_p' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$, choose $u_1'y_1 + u_2'y_2 = 0 \implies y_p' = u_1y_1' + u_2y_2' \implies y_p'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$
- Subbing it back into the DE: $$\begin{aligned} & (u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2') + a(u_1y_1' + u_2y_2') + b(u_1y_1 + u_2y_2) = \phi(x) \\ \implies & (y_1'' + ay_1' + by_1)u_1 + (y_2'' + ay_2' + by_2)u_2 + u_1'y_1' + u_2'y_2' = \phi(x) \\ \implies & u_1'y_1' + u_2'y_2' = \phi(x) \end{aligned}$$

- $\begin{cases} u_1'y_1' + u_2'y_2' = \phi(x) \\ u_1'y_1 + u_2'y_2 = 0 \end{cases}$ has 2 equations and 2 unknowns and allows us to solve for u_1 and u_2
- To solve this in general: $u_1' = -\frac{y_2\phi}{y_1y_2' - y_2y_1'}, u_2' = \frac{y_1\phi}{y_1y_2' - y_2y_1'}$

- Example: $y'' + y' - 2y = e^x$
  - $r^2 + r - 2 = 0 \implies r_1 = -2, r_2 = 1 \implies y_c = C_1e^x + C_2e^{-2x} \implies y_1 = e^x, y_2 = e^{-2x}$
  - Impose $\begin{cases} u_1'e^x - 2u_2'e^{-2x} = e^x \\ u_1'e^x + u_2'e^{-2x} = 0 \end{cases}$
  - $u_2'e^{-2x} = -u_1'e^x \implies u_1'e^x + 2u_1'e^x = e^x \implies 3u_1' = 1 \implies u_1 = \frac{1}{3}x$
    * Note that we don't need to worry about the constant of integration since this would simply become a part of the constant in y_c
  - $u_2'e^{-2x} = -\frac{1}{3}e^x \implies u_2' = -\frac{1}{3}e^{3x} \implies u_2 = -\frac{1}{9}e^{3x}$
  - $y_p = \frac{1}{3}xe^x - \frac{1}{9}e^x \implies y = C_1e^x + C_2e^{-2x} + \frac{1}{3}xe^x - \frac{1}{9}e^x = C_1e^x + C_2e^{-2x} + \frac{1}{3}xe^x$
- Example: $y'' + y = 3\sin x \sin(2x)$
  - ϕ does not lead to an obvious trial solution, so using the method of undetermined coefficients is hard in this case
  - $r^2 + 1 = 0 \implies r = \pm i \implies y_c = A\cos x + B\sin x$
  - Let $y_p = u_1(x)\cos x + u_2(x)\sin x$
  - $\begin{cases} u_1'\cos x + u_2\sin x = 0 \\ -u_1'\sin x + u_2'\cos x = 3\sin x\sin(2x) \end{cases}$
  - $$\begin{aligned} u_1' &= -\frac{y_2\phi}{y_1y_2' - y_2y_1'} \\ &= -\frac{-3\sin^2 x\sin(2x)}{\cos^2 x + \sin^2 x} \\ &= -3\sin^2 x\sin(2x) \end{aligned}$$
  - $$\begin{aligned} u_1 &= \int -3\sin^2 x\sin(2x)\,\mathrm{d}x \\ &= -3\int \sin^2 x(2\sin x\cos x)\,\mathrm{d}x \\ &= -6\int \sin^3 x\cos x\,\mathrm{d}x \\ &= -6\int v^3\,\mathrm{d}v \\ &= \frac{-3}{2}v^4 \\ &= \frac{-3}{2}\sin^4 x \end{aligned}$$
  - $u_2' = 3\cos x\sin x\sin(2x)$
  - $$\begin{aligned} u_2 &= \int 3\cos x\sin x\sin(2x)\,\mathrm{d}x \\ &= 3\int \left(\frac{1}{2}\sin(2x)\right)\sin(2x)\,\mathrm{d}x \\ &= \frac{3}{2}\int \sin^2(2x)\,\mathrm{d}x \\ &= \frac{3}{2}\int \frac{1}{2}(1 - \cos(4x))\,\mathrm{d}x \\ &= \frac{3}{4}\left(x - \frac{1}{4}\sin(4x)\right) \end{aligned}$$

- $y_p = -\frac{3}{2}\cos x \sin^4 x + \frac{3}{16}(4x - \sin(4x))\sin x \implies y = -\frac{3}{2}\cos x \sin^4 x + \frac{3}{16}(4x - \sin(4x))\sin x + A\cos x + B\sin x$