Lecture 8, Oct 6, 2021

Matrices as Linear Transformations

- Projections can be written as the product of a vector and a projection matrix
- Generally, all linear transformations can be expressed as a matrix; to determine the matrix associated with a linear transformation, find $\mathcal{L}(\hat{\imath})$, $\mathcal{L}(\hat{\jmath})$, $\mathcal{L}(\hat{k})$ (and so on for all bases) and put those as the columns of the matrix
  - Consider $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = v_1\vec{b}_1 + v_2\vec{b}_2 + \cdots$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, etc
  - By linearity $\mathcal{L}(\vec{v}) = v_1\mathcal{L}(\vec{b}_1) + v_2\mathcal{L}(\vec{b}_2) + \cdots$
  - Now consider the matrix-vector product $\begin{bmatrix} | & | & | \\ \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \cdots \\ | & | & | \end{bmatrix} \vec{v}$
  - By definition this is equal to $v_1\mathcal{L}(\vec{b}_1) + v_2\mathcal{L}(\vec{b}_2) + \cdots$ which is equal to $\mathcal{L}(\vec{v})$
  - Therefore $\mathcal{L}$ is represented by the matrix $\begin{bmatrix} | & | & | \\ \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \cdots \\ | & | & | \end{bmatrix}$, so every linear transformation has an associated matrix
- Conversely all matrix transformations are also linear
- Some famous transformations:
  - I – The identity transformation, which takes every vector to itself; the identity matrix has 1s down the diagonal and 0s everywhere else
  - O – The zero transformation, which takes every vector to zero; the zero matrix has all zeroes

Why We Multiply Matrices the Way We Do

- Let T_1 and T_2 be two linear transformations
- The transformation of T_1 followed by T_2 is $T_2(T_1(\vec{v}))$; note order is important here
- $T_2(T_1(\vec{v})) = M_2(M_1\vec{v})$ where M_2 is the matrix for transformation T_2 and M_1 is the matrix for transformation T_1
- Therefore the matrix product M_2M_1 represents the result of applying two linear transformations, one after the other
- $A \begin{bmatrix} | & | & | \\ \vec{b_1} & \vec{b_2} & \cdots \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\vec{b_1} & A\vec{b_2} & \cdots \\ | & | & | \end{bmatrix}$
- Example: Define a transformation T_θ that takes every vector in $\mathbb{R}^2$ and rotates it counterclockwise by θ; the matrix for this transformation is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$