Lecture 6, Sep 29, 2021

Solving Linear Equations

- Solving systems of linear equations is a central problem of linear algebra
- Row picture: Visualize every row as a line or plane in space, and find the intersection of every row
- Column picture: Express the system as a single vector equation, and find the correct linear combination of vectors that satisfy the equation:
  - e.g. $\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \implies x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$
- Matrix form: $A\vec{x} = \vec{b}$
  - The matrix vector product on the left hand side is defined to be the vector given by $A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{a_1} & \vec{a_2} & \cdots \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots$
  - This leads to the dot product rule of calculating a matrix-vector product: Each entry in the resulting vector is the dot product of one row of A and $\vec{x}$

What is a Matrix?

- A rectangular array of numbers: $A = \begin{bmatrix} 4 & 8 & 3 \\ 2 & 1 & 9 \end{bmatrix}$, a 2×3 matrix
- Matrix size is rows $\times$ columns
- More general notation: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Matrix Operations

- Matrix addition: Just add all the corresponding entries
  - Requirement: Two matrices have the same size
  - Commutative, associative
- Scalar multiplication: Multiply every entry by the scalar
- Matrix multiplication:
  - The entry at row i, column j is the dot product of row i of A with column j of B
  - $A \begin{bmatrix} | & | & | \\ \vec{b_1} & \vec{b_2} & \cdots \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\vec{b_1} & A\vec{b_2} & \cdots \\ | & | & | \end{bmatrix}$
  - Requirement: The first matrix has the same number of columns as rows of the second matrix: $m \times n$ can multiply $n \times k$ to produce a $m \times k$ matrix
  - **Not commutative**, but associative
  - The notation for multiplying matrices is just AB, without any symbol in between
- Properties:
  1. $A + B = B + A$ addition is commutative
  2. $c(A + B) = cA + cB$ scalar multiplication is distributive
  3. $(A + B) + C = A + (B + C)$ addition is associative
  4. $C(A + B) = CA + CB$ (note C is a matrix) matrix multiplication is distributive
  5. $A(BC) = (AB)C$ matrix multiplication is associative
  6. All exponent laws apply to matrices (e.g. $A^pA^q = A^{p+q}$)